THE

PUPIL TEACHERS'

COURSE OF

MATHEMATICS

PART I.

THE PUPIL-TEACHER'S COURSE

OF

MATHEMATICS.

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PART I.

EUCLID, BOOKS I. & II.

WITH

Notes, Examples, and Explanations.

BY

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LONDON:

NATIONAL SOCIETY'S DEPOSITORY,

WESTMINSTER.

1879.

PREFACE.

I HAVE long thought that a course of Mathematics for Pupil-teachers, covering the ground they are required to traverse, with hints and notes on points at which beginners find a difficulty, is urgently needed. Their course includes the first two books of Euclid, Algebra to the end of quadratic equations, and the mensuration of plane surfaces, and this, the first part of the course, contains two books of Euclid. No doubt there are good editions of Euclid, but in the first place they comprehend several Books and are therefore unnecessarily bulky and expensive for pupil-teachers ; in the second place, the notes and explanations are for the most part adapted to a more advanced class of students (for instance, in Pott's excellent edition), and not full enough for pupil-teachers, who have many other subjects to learn at the close of days devoted to teaching; and again, many of the examples (deductions) are too hard for them and are not graduated sufficiently. I

have examined for the Oxford and Cambridge
Board, the Cambridge Syndicate, &c., a large
number of our great public and middle-class
schools, and have also awarded the certificates of
the Board in Mathematics, and it is surprising how
frequently the same errors and a similar confusion
of ideas recur: against these I have endeavoured
to guard in the notes and hints. It is an im-
portant question in editing a treatise on geometry,
whether abbreviations may be allowed. I have
followed our rule at the Cambridge Local Ex-
aminations, that, whilst no symbols *of operation*
(such as −, +, ×) are admissible, all generally
understood abbreviations for *words* may be used.
The question is important, for *magnitude* not
number is the subject of geometry, and it there-
fore would be contrary to strict reasoning on space
to introduce symbols which refer distinctly to
operations of quantity. Moreover, apart from the
illogical position of such symbols in geometry,
nothing so much causes a beginner to confuse
together algebraical and geometrical notions as the
connection of these symbols with geometrical
magnitudes. But whilst I recognise the truth
and importance of this, there seems no advantage
in forcing a student to write out words at full
length which occur several times in propositions ;
and I have therefore introduced abbreviations for

certain words, of which a list will be given. The examples are attached to the propositions on which they respectively mainly depend : most of them have been set at examinations of schools, and several at the monthly collective examinations of pupil-teachers. The learner should by no means be satisfied with mastering the propositions : the real test of geometrical knowledge is ability to work problems, and therefore at every examination great weight is given to this point. With patience and careful thought every boy of fair ability will be able to solve many of the deductions I have given, and when he has done so, his future progress will be comparatively easy. The beginner is recommended to study thoroughly the propositions to the end of the 26th (which is the course for pupil-teachers at the end of the third year), and then to begin again, working out the *riders*, as deductions are termed when they are attached to propositions by aid of which they may be solved. A rider is therefore in one respect much easier than a deduction set as a separate and independent example, inasmuch as the student knows that the parent proposition is the key to its solution. In another point of view, however, it is more difficult, since in working it, only the parent and any of the preceding propositions may be appealed to, whereas in the case of a deduction

the field is more open, since (within certain limits) any propositions may be used in the demonstration.

Although this little book is primarily intended for pupil-teachers, it is hoped that it may also be found useful by students of the first year at Training Colleges, and by others who are commencing the study of Geometry.

January 1879.

PUPIL-TEACHER'S

COURSE OF MATHEMATICS.

PART I.

EUCLID, BOOKS I. II.

BOOK I.

DEFINITIONS.

1. A point is that which has no parts, or which has no magnitude.

[Thus a point has place (position), but no size (magnitude) : it cannot be parted or divided.]

2. A line is length without breadth.

3. The extremities of lines are points.

[It is clear that the intersections of lines are also points.]

4. A straight line is that which lies evenly between its extreme points.

[When the direction of a line is known, the line is said to be given in position; also when its length is known, it is said to be given in magnitude.]

5. A superficies [or surface] is that which has only length and breadth.

6. The extremities of a superficies are lines.

B

7. A plane superficies is that in which any two points being taken, the straight line between them lies wholly in that superficies.

[All these seven definitions express abstract ideas (you will remember the force of the word *abstract* in the term abstract noun). Every visible line, for instance, has both length and breadth : we cannot draw a line which has no breadth, but we can reason about lines as having an independent existence, and this is what Euclid requires us to do.]

8. A plane angle is the inclination of two lines to one another in a plane, which meet together, but are not in the same direction.

[This includes the angles formed by two curved lines, as well as that formed by two straight lines, and is not required in the first two books.]

9. A plane rectilineal angle is the inclination of two *straight* lines to one another, which meet together, but are not in the same straight line.

[Thus the inclination of the lines OA, OB is called the angle AOB, or BOA, or the angle at O.
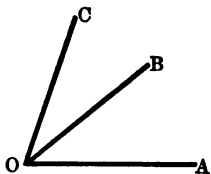
Observe that the magnitude of an angle is entirely independent of the *lengths* of the two lines which form it, and no change is made in the angle by making these lines longer or shorter. The *inclination* of the lines to one another, the amount of opening between them, is the measure of its magnitude. It is not very uncommon for a beginner, when asked by his teacher 'If from the angle COA you take the angle BOA, what angle remains?' to reply, 'If you take away

the angle BOA, all that remains is the line CO.' Now this is a complete mistake, for angles and lines are altogether different things. If an angle be taken from an angle, it is not a line that remains but an angle (viz. in this case the angle COB), just as when you take money from money, the remainder is money, not ounces or yards. The point at which the lines meet is often called the vertex of the angle.]

10. When a straight line standing on another straight line makes the adjacent angles equal to one another, each of these angles is called a right angle; and the straight line which stands on the other is called a perpendicular to it.

11. An obtuse angle is that which is greater than a right angle.

12. An acute angle is that which is less than a right angle.

13. A term or boundary is the extremity of anything.

14. A figure is that which is enclosed by one or more boundaries.

15. A circle is a plane figure contained by one line, which is called the circumference, and is such that all

straight lines drawn from a certain point within the figure to the circumference, are equal to one another.
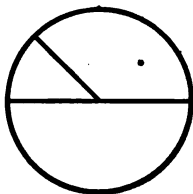
16. And this point is called the centre of the circle.

17. A diameter of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

[A radius is a straight line drawn from the centre to the circumference, and is therefore half the diameter.]

18. A semicircle is the figure contained by a diameter and the part of the circumference cut off by the diameter.

19. The centre of a semicircle is the same as that of the circle.

20. Rectilineal figures are those which are contained by straight lines.

21. Trilateral figures, or triangles, by three straight lines.

22. Quadrilateral figures, by four straight lines.

23. Multilateral figures, or polygons, by more than four straight lines.

[A polygon is called *regular* when all its sides are equal and all its angles are also equal.]

24. Of three-sided figures :

An equilateral triangle is that which has three equal sides.
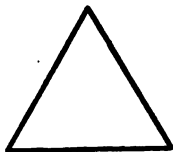
25. An isosceles triangle is that which has two sides equal.
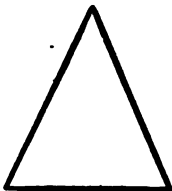
26. A scalene triangle is that which has three unequal sides.

27. A right-angled triangle is that which has a right angle.

[The side opposite the right angle is often called the hypotenuse.]

28. An obtuse-angled triangle is that which has an obtuse angle.
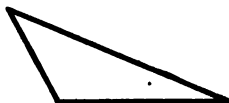
[It will be seen hereafter that a triangle can only have one right angle, or one obtuse angle.]

29. An acute-angled triangle is that which has three acute angles.

[In any triangle the word *base* is often applied to any one of the sides to distinguish it from the other two, and the angular point opposite to that side is called the *vertex*.]

30. Of four-sided figures :

A square is that which has all its sides equal, and all its angles right angles.
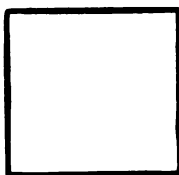
31. An oblong is that which has all its angles right angles, but not all its sides equal.

32. A rhombus has all its sides equal, but its angles are not right angles.
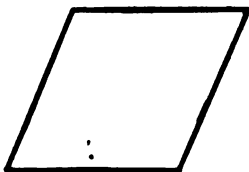
33. A rhomboid has its opposite sides equal to one another, but all its sides are not equal, nor its angles right angles.

34. All other four-sided figures besides these are called trapeziums.

[Four-sided figures which have two opposite sides parallel are often called trapezoids.]

35. Parallel straight lines are such as are in the same plane, and which being produced ever so far both ways do not meet.
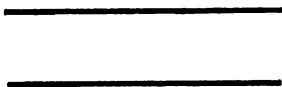
36. A parallelogram is a four-sided figure, of which the opposite sides are parallel.

37. The diameter (or diagonal) of a parallelogram is the straight line joining two of its opposite angles.

[At the close of the definitions Euclid inserts three Postulates, that is, three things which he asks to be allowed to do (from *postulare*, to ask), and twelve axioms, that is, statements which he claims to be taken as true without proof, from a Greek word meaning to claim. These will now be given, and some remarks will be made on their nature afterwards.]

POSTULATES.

Let it be granted,

1. That a straight line may be drawn from any one point to any other point.

2. That a terminated straight line may be produced to any length in a straight line.

3. That a circle may be described from any centre, at any distance from that centre.

AXIOMS.

1. Things which are equal to the same thing are equal to one another.

2. If equals be added to equals the wholes are equal.

3. If equals be taken from equals the remainders are equal.

4. If equals be added to unequals the wholes are unequal.

5. If equals be taken from unequals the remainders are unequal.

6. Things which are double of the same thing are equal to one another.

7. Things which are halves of the same thing are equal to one another.

8. Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another.

9. The whole is greater than its part.

10. Two straight lines cannot enclose a space.

11. All right angles are equal to one another.

12. If a straight line meet two straight lines, so as to make the interior angles on the same side of it taken together less than two right angles, these straight lines,

being continually produced, shall at length meet on that side on which are the angles which are less than two right angles.

[Geometrical propositions (things proposed) are of two kinds, *problems* and *theorems*. In a problem some geometrical construction is required to be made, as ' to describe an equilateral triangle on a given finite straight line.' In a theorem, some asserted geometrical property is to be demonstrated (or shown true), as 'the angles at the base of an isosceles triangle are equal to one another.' Thus in a problem there are data (things given) and quæsita (things required) and a solution is sought ; in a theorem there are the hypothesis (supposition or things admitted), and the conclusion (things to be shown true), and a demonstration or proof is required. In the example of a problem just given, the thing given is the straight line, the thing required (the construction to be made) is an equilateral triangle upon it. In the example of a theorem the hypothesis is that the triangle is isosceles, the conclusion (what has to be shown true) is the statement that the base angles are equal.

The postulates are problems of which the possibility is admitted without proof, as self-evident ; similarly the axioms are theorems or statements which cannot be made more evident by demonstration.

Every proposition consists of the following parts : first, there is the general enunciation ; then the particular enunciation, that is, the previous general enunciation repeated, but with reference to and explained by the diagram. Next the construction, in which certain lines, which are necessary to do what has to be done, or to prove what has to be proved, are drawn.* Then follows the demonstration, showing the truth or falsehood of the theorem, or the possibility or impossibility of the problem. The demonstration is said to be *direct* when the conclusion is obtained directly from the hypothesis. It is *indirect* when it is proved that the introduction of any other supposition leads to an absurdity: in other words, when the conclusion is obtained by showing that some absurdity follows from supposing the required result to be untrue. This method is called the ' *reductio ad absurdum.*'

* In some cases no construction is required ; see, for instance, note to Prop. 15.

The beginner will understand these distinctions and divisions more clearly when he has learnt a few propositions and has separated each into its different parts. When they are once mastered, his task will become very much easier.

I now proceed to give the abbreviations which will be allowed in the first book.

pt.	for point	∠	for angle	
rt.	„ right	△	„ triangle	
st.	„ straight	=	„ is (or are) equal to, or, equal to	
sq.	„ square			
int.	„ interior	‖	„ parallel	
ext.	„ exterior	⊙	„ circle	
rectil.	„ rectilinear	⊙ce	„ circumference	
equilat.	„ equilateral	⊥	„ perpendicular	
rem.	„ remaining, or remainder	▱	„ parallelogram	
		∵	„ because	
adj.	„ adjacent	∴	„ therefore.	
opp.	„ opposite			

It is usual to place the letters Q.E.F. (an abbreviation for *quod erat faciendum*, that is, *which was to be done*) at the end of the discussion of a problem, and the letters Q.E.D. (an abbreviation for *quod erat demonstrandum*, that is, which was to be demonstrated, or proved), at the end of the discussion of a theorem. Hyp. stands for hypothesis, constr. for construction, dem. for demonstration.]

BOOK I. SECTION I.

(PROPOSITIONS I.—XXVI.)

PROPOSITION I. PROBLEM.

To describe an equilateral triangle on a given finite straight line.

Let AB be the given st. line. It is required to describe an equilat. △ on AB.
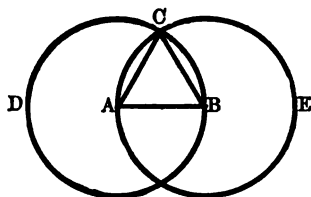
Constr. From the centre A, at the distance AB, describe

  ⊙ BCD ;.   (post. 3)

from the centre B, at the distance BA, describe

  ⊙ ACE ;   (post. 3)

and from the pt. C, in which the ⊙s cut one another,

  draw the st. lines CA,CB, to the pts. A,B.   (post. 1)

            Then ABC shall be an equilat. △.

Dem. For, ∵ the pt. A is the centre of ⊙ BCD,

  ∴ AC = AB,   (def. 15)

       and ∵ the pt. B is the centre of ⊙ ACE,

               ∴ BC = AB ;

     but it has been proved that AC = AB ;

      ∴ AC, BC, are each of them = AB ;

but things which are equal to the same thing are equal to
one another ;   (ax. 1)

$$\therefore AC = BC;$$

Thus AB, BC, AC are equal to one another :
and $\triangle$ ABC is therefore equilateral,   (def. 24)
and it is described on the given straight line AB.   Q. E. F.

[Here the *data* are the given st. line ; the *quæsita,* an
equilateral triangle on it.   *The general enunciation,* 'to de-
scribe an equilateral triangle on a given finite straight line ;'
the *particular enunciation,* 'let AB be the given st. line.   It
is required to describe an equilat. $\triangle$ on AB.'   It may be
observed that in Books i. and ii. no properties of the circle
are discussed in any way ; it is merely employed in con-
structions, as defined in def. 15, and as granted by the third
postulate.]

By a method similar to that used in this problem,
describe on a given finite straight line an isosceles tri-
angle, whose sides shall be each double of the base.

PROPOSITION II. PROBLEM.

From a given point, to draw a straight line equal to a given straight line.

Let A be the given pt., and BC the given straight line. It is required to draw from the pt. A a st. line equal to BC.
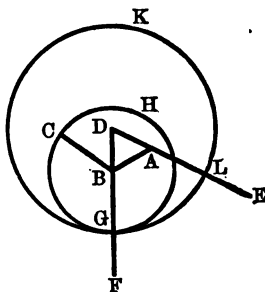
Constr. From the pt. A to B draw the st. line AB ;

(post. 1)

on AB describe the equilat. △ ABD,   (i. 1)

and produce the st. lines DA, DB to E and F ; (post. 2)

from the centre B, at the distance BC, describe ⊙ CHG ;

(post. 3)

and from the centre D, at the distance DG, describe ⊙ GKL.   (d°)

Then the st. line AL shall be = BC.

Dem. ∵ the pt. B is the centre of ⊙ CHG,

∴ BC = BG ;   (def. 15)

and ∵ D is the centre of ⊙ GKL,

∴ DL = DG   (d°)

and DA, DB parts of them are equal ; (def. 24)

∴ the rem. AL = the rem. BG ;   (ax. 3)

but it has been shown that BC = BG,

∴ AL and BC are each of them equal to BG ;
and things that are equal to the same thing are equal to
one another ;   (ax. 1)

∴ AL = BC.

Wherefore, *from the given pt. A a st. line AL has been
drawn equal to the given st. line BC.*   Q. E. F.

[It is unnecessary to point out the *general enunciation*,
and *particular enunciation* ; the beginner will easily be able
to do this for himself, from the explanation given.   The data
are the given point and the given straight line ; the quæsita,
a straight line equal to the given one.   Observe how this pro-
position depends on the first (the side reference i. 1, means
Book i., prop. 1) ; the learner will find that throughout Euclid
the successive propositions are linked to and depend upon
the preceding ones, as the successive links of a chain hang
one from another.

When the given point is neither in the line nor in the
line produced, there are several ways in which the line may
be drawn.   For (1) the given line has two ends, each of
which may be joined to the given point ; (2) the equilateral
triangle may be described on either side of this line ; and
(3) the side BD may be produced either way.   It will be an
excellent exercise to draw figures and to work out the pro-
position for each case : this will give eight different examples
of this problem.]

PROPOSITION III.   PROBLEM.

From the greater of two given straight lines to cut off a part equal to the less.

Let AB and CD be the two given st. lines, of which AB is the greater.

It is required to cut off from AB, the greater, a part equal to CD, the less.
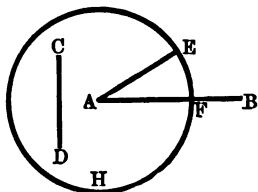
Constr. From the pt. A draw the st. line AE = CD; (i. 2)
  with centre A at the distance AE describe ⊙ EFH,
                                                    (post. 3)
              cutting AB in the pt. F.
              Then will AF = CD.
Dem.   ∵ A is the centre of ⊙ EFH,
                    ∴ AF = AE ;   (def. 15)
          but the st. line CD = AE ; (constr.)
    whence AF and CD are each of them equal to AE ;
                    ∴ AF = CD.   (ax. 1)
  Wherefore, *from AB, the greater of two st. lines, a part AF has been cut off equal to CD, the less.*   Q. E. F.

[The data are two given st. lines, the quæsita, to cut off from the greater a part equal to the less.]

PROPOSITION IV. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles contained by those sides equal to one another, they shall also have their bases or third sides equal; and the two triangles shall be equal, and their other angles shall be equal, each to each, namely those to which the equal sides are opposite.

Let ABC, DEF be two △s which have the two sides AB, AC equal to the two sides DE, DF, each to each, namely AB to DE, and AC to DF, and the included ∠ BAC equal to the included ∠ FDF.

Then shall the base BC be equal to the base EF, and △ ABC to △ DEF, and the other ∠s, to which the equal sides are opposite, shall be equal, each to each, namely ∠ ABC to ∠ DEF, and ∠ ACB to ∠ DFE.
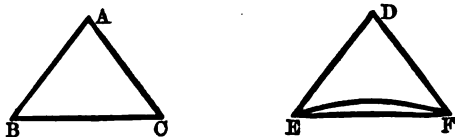
Dem.   For if △ ABC be applied to △ DEF,
So that the pt. A may be on D, and the st. line AB on
   DE ;
$$\because AB = DE,   \text{(hyp.)}$$
         ∴ the pt. B shall coincide with the pt. E ;
and AB coinciding with DE, $\because$ ∠ BAC = ∠ EDF, (hyp.)
            ∴ the st. line AC shall fall on DF ;
               also $\because$ AC = DF,
            ∴ the pt. C shall coincide with F ;
               but the pt. B coincides with E ;
      ∴ the base BC shall coincide with the base EF ;

for, the pt. B coinciding with E, and C with F, if the
base BC do not coincide with the base EF, the two st.
lines BC and EF would enclose a space, which is im-
possible.   (ax. 10)

∴ the base BC does coincide with the base EF, and is
  equal to it.   (ax. 8)

Wherefore the whole △ ABC coincides with the whole
  △ DEF, and is equal to it ;   (ax. 8)

and the other ∠ s of the one coincide with the other ∠ s
  of the other, and are equal to them, namely ∠ ABC to
  ∠ DEF, and ∠ ACB to ∠ DFE.

  Therefore, *if two triangles*, &c.   Q. E. D.

[The *hypothesis* in this, the first theorem, is that two △s
have two sides and the included ∠ of the one equal to two
sides and the included ∠ of the other, each to each ; the
conclusion is that their bases shall be equal, &c. The eighth
axiom contains the principle of superposition, by which
we conceive one figure to be placed on another, and prove
that their boundaries exactly coincide, and we thence infer
that the figures are in all respects equal. It may be re-
garded as the definition of geometrical equality, and it is
first employed in this proposition.]

  1. A straight line bisects the vertical angle of an
isosceles triangle ; show that it also bisects the base.

  2. ABDE, BFGC are squares on two sides of the
triangle ABC, and AF, CD are joined ; show that AF, CD
are equal.

  3. Two st. lines bisect one another at right angles ;
show that any point in either of them is equidistant from
the extremities of the other.

PROPOSITION V.   THEOREM.

The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced, the angles on the other side of the base shall be equal.

Let ABC be an isosceles △ in which AB=AC, and let the equal sides AB, AC be produced to D and E.

Then shall ∠ ABC = ∠ ACB,

and ∠ DBC = ∠ ECB.
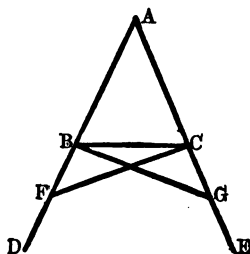
Constr. In BD take any pt. F ;

from AE the greater, cut off AG = AF the less,   (i. 3)

join FC, GB.

Dem. ∵ AF = AG, (constr.) and AB = AC ;   (hyp.)

the two sides FA, AC = the two GA, AB, each to each, and they contain the ∠ FAG common to the two △s AFC, AGB ;

∴ base FC = base GB,

and △ AFC = △ AGB ;   (i. 4)

also the rem. ∠ s of the one = the rem. ∠ s of the other, each to each, to which the equal sides are opposite ;

(i. 4)

namely ∠ ACF = ∠ ABG, and ∠ AFC = ∠ AGB.

And ∵ the whole AF = the whole A G,

of which the part AB = the part AC,   (hyp.)

∴ the remainder BF = the remainder CG ; (ax. 3)
and it was proved that FC=GB ;
hence ∵ the two sides BF, FC = the two CG, GB, each
to each, and it was proved that ∠ BFC = ∠ CGB,
∴ △s BFC, CGB are equal ; (i. 4)
and their other ∠ s, each to each, to which the = sides
are opp.; namely ∠ FBC=∠ GCB, and ∠ BCF=∠ CBG.
(i. 4)

And since it has been demonstrated that
∠ ABG = ∠ ACF,
the parts of which, ∠ s CBG, BCF are also = ;
∴ rem. ∠ ABC = rem. ∠ ACB, (ax. 3)
which are ∠ s at the base of △ ABC ;
and it has been proved that ∠ FBC = ∠ GCB, which
are ∠ s on the other side of the base.

Therefore, *the angles at the base*, &c. Q. E. D.

Cor. Hence every equilateral triangle is also equi-
angular.

[Hypothesis, an isosceles △ ; conclusion that ∠ s at the
base are equal, and if the equal sides be produced, the ∠ s
on the other side of the base shall be equal.

Observe, the construction is simple enough, and the
demonstration falls naturally into three parts : (1) △s
ABG,ACF are proved equal by prop. 4, (2) △s BCF,BCG
are also proved equal by prop. 4, (3) ∠ s ABC,ACB are
proved equal by ax. 3.

A *corollary* (cor.) is an inference which at once follows
from the demonstration of a proposition : in fact, it is a simple
'rider' (see preface). The student should not leave it
without seeing clearly why it so follows.]

1. The middle point of the base of an isosceles tri-
angle is joined to the vertex : show that the triangles so
formed are equal in all respects.

2. The straight line which bisects the base of an
isosceles triangle at right angles shall pass through the
vertex.

PROPOSITION VI. THEOREM. ·

If two angles of a triangle be equal to one another, the sides also which subtend, or are opposite to, the equal angles, shall be equal to one another.

Let ABC be △ in which ∠ ABC = ∠ ACB.
Then shall AB = AC.
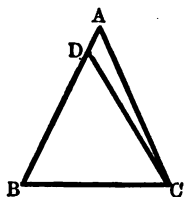
Constr. For, if AB be not equal to AC, one of them is greater than the other. Let AB be greater than AC ; and at the pt. B, from BA cut off BD = CA the less,

<div align="right">(i. 3)</div>

and join DC.

Dem. Then in △s DBC, ABC, ∵ DB = AC and BC is common,

The two sides DB, BC = the two sides AC, CB, each to each ;

and ∠ DBC = ∠ ACB ;   (hyp.)

∴ base DC = base AB, and △ DBC = △ ACB,   (i. 4)

the less = the greater, which is absurd.   (ax. 9)

∴ AB is not unequal to AC, that is, AB = AC.

Wherefore, *if two angles*, &c.   Q. E. D.

Cor. Hènce every equiangular triangle is also equilateral.

[Hypothesis, two angles of a triangle are equal. Conclusion, the opposite sides are also equal. This is the converse of the first part of prop. 5. One prop. is said to ~ ~ the converse of another when the conclusion of each is

the hypothesis of the other. In prop. 5 the hypothesis is the equality of the sides, and the conclusion is the equality of the base angles. In prop. 6 the hypothesis is the equality of the base angles, and the conclusion the equality of the sides. Converse propositions are not universally true : an example of this will be given in the note to prop. 8. This is the first proposition in which we find an *indirect* (see note before prop. 1) proof, and it may be observed that few converse propositions admit of a direct proof.]

The angles at the base of an isosceles triangle are bisected : show that the bisecting lines form with the base another isosceles triangle.

PROPOSITION VII.   THEOREM.

On the same base, and on the same side of it, there cannot be two triangles that have their sides which are terminated at one extremity of the base equal to one another, and likewise those which are terminated at the other extremity equal to one another.

If it be possible, on the same base AB, and on the same side of it, let there be two △s ACB, ADB, which have their sides CA, DA, terminated at the extremity A of the base, equal to one another, and likewise their sides CB, DB, that are terminated at B, equal to one another.
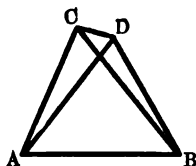
First. When the vertex of each of the △s is without the other △.

Constr.   Join CD.

Dem.   ∵ AC = AD ∴ ∠ ACD = ∠ ADC ;   (i. 5)
          but ∠ ACD is greater than ∠ BCD :   (ax. 9)
          ∴ also ∠ ADC is greater than ∠ BCD ;
       much more then is ∠ BDC greater than ∠ BCD.
       Again, ∵ BC = BD, (hyp.) ∴ ∠ BDC = ∠ BCD ;
                                                   (i. 5)
          but ∠ BDC was proved greater than ∠ BCD,
hence ∠ BDC is both equal to, and greater than ∠ BCD;
                    which is impossible.

Secondly. Let the vertex D of △ ADB fall within ACB.
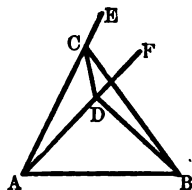
Constr. Join CD, produce AC to E, and AD to F.

Dem. Then ∵ AC = AD in △ ACD, ∴ ∠s ECD, FDC on the other side of the base CD are equal to one another ;   (i. 5)

      but ∠ ECD is greater than ∠ BCD ;   (ax. 9)

    ∴ also ∠ FDC is greater than ∠ BCD ;

  much more then is ∠ BDC greater than ∠ BCD.

      Again, ∵ BC = BD ∴ ∠ BDC = ∠ BCD ; (i. 5) wherefore ∠ BDC is both equal to and greater than ∠ BCD ; which is impossible.

Thirdly. The case in which the vertex of one triangle is on a side of the other needs no demonstration.

      Therefore, *on the same base,* &c.   Q. E. D.

[Hypothesis : two triangles on the same base and on the same side of it. Conclusion : they cannot have their sides which are terminated at one extremity of the base equal to one another, and likewise those which are terminated at the other extremity. This proposition is required directly only in the proof of prop. 8 ; *indirectly* however it is often required, inasmuch as prop. 8 (which depends on it) is frequently made use of.]

PROPOSITION VIII.   THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal; the angle which is contained by the two sides of the one shall be equal to the angle contained by the two sides equal to them, of the other.

Let ABC, DEF, be two △s, having the two sides AB, AC = the two sides DE, DF, each to each, namely AB to DE, and AC to DF, and also the base BC = base EF.

Then shall ∠ BAC = ∠ EDF.
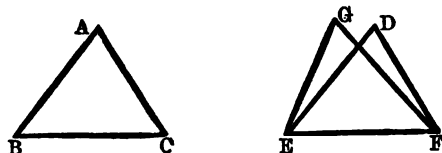
Dem. For if △ ABC be applied to △ DEF, so that pt. B be on E, and st. line BC on EF ;

then ∵ BC = EF, (hyp.) ∴ C shall coincide with F ;

wherefore, BC coinciding with EF, BA and AC shall coincide with ED, DF ;

for, if the base BC coincide with the base EF, but the sides BA, AC do not coincide with the sides ED, DF, but have a different situation as EG, FG ;

Then, on the same base and on the same side of it there can be two △s which have their sides which are terminated at one extremity of the base equal to one another, and likewise their sides which are terminated at the other extremity ; but this is impossible.   (i. 7)

Therefore, if the base BC coincide with the base EF, the sides BA, AC cannot but coincide with the sides

ED, DF ; wherefore also ∠ BAC coincides with ∠ EDF, and is equal to it. . (ax. 8)

Therefore, *if two triangles*, &c. Q. E. D.

[Hypothesis, two triangles having two sides of the one equal to two sides of the other, and also their bases equal. Conclusion, the angle contained by the two sides of the one shall be equal to the angle contained by the two sides of the other. The △s having been proved to coincide are mani- · festly equal in all respects. Observe however this is not stated in the conclusion, and in subsequent propositions Euclid uses prop. 4 when he requires more than this. The converse proposition would run thus : ' If two △s have the 3 ∠ s in the one equal to the 3 ∠ s in the other, each to each, the 3 sides shall be equal, each to each.' The beginner may at once satisfy himself that this is not true by looking at this figure]:

1. From every point of a given line the lines drawn to each of two given points on opposite sides of the line are equal ; prove that the line joining the given points will be bisected by the given line at right angles.

2. If two circles cut each other, the line joining their points of intersection is bisected at right angles by the line joining their centres.

PROPOSITION IX.   PROBLEM.

To bisect a given rectilineal angle, that is, to divide it into two equal parts.

Let BAC be the given rectlin. $\angle$.
It is required to bisect it.
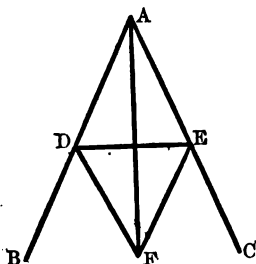
Constr.   Take any pt. D in AB ;
      from AC cut off AE = AD, (i. 3) and join DE,
on the side of DE remote from A, describe an equilat.
  $\triangle$ DEF, (i. 1) and join AF.
          Then AF shall bisect $\angle$ BAC.
Dem. $\because$ AD = AE (constr.), and AF is common to the
  $\triangle$s DAF, EAF ;
the two sides AD, AF = the two sides EA, AF, each to
  each ;
and base DF=base EF, (def. 24) $\therefore$ $\angle$ DAF= $\angle$ EAF.
                                        (i. 8)
     Wherefore, *the angle BAC is bisected by the straight
line AF.*   Q. E. F.

[The learner should here, and in all following problems,
find out for himself the *data* and *quæsita*.   The equilat. $\triangle$
is described on the side *remote from A* because the three
lines DA, AE, EF might possibly be all equal, that is DAE
might itself be an equilat. $\triangle$.   In this case if the equilat.

△ DFE were described on the same side of DE on which A is, the vertex F would fall exactly on A, and there would be no line AF. Also if the ∠ BAC were such that it would fall within the equilat. △ DEF when described on the same side of DE as A, the line AF would not in this case bisect ∠ BAC, and if we produced FA to some pt. G we do not at present know that ∠ DAG = ∠ EAG : this is proved by prop. 13. Observe that by help of this prop. an ∠ may be divided into 4, 8, 16 &c. equal ∠ s.]

2

PROPOSITION X. PROBLEM.

To bisect a given finite straight line, that is, to divide it into two equal parts.

Let AB be the given finite st. line.

It is required to divide AB into two equal parts.
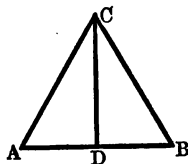
Constr. On AB describe the equilat. △ ABC ;   (i. 1)

      and bisect ∠ ACB by CD meeting AB at D. (i. 9)

         Then AB shall be bisected at D.

Dem. ∵ AC = BC, (def. 24) and CD is common to △s ACD, BCD ;

   the two sides AC, CD = BC, CD each to each ;

and ∠ ACD = ∠ BCD ; (constr.) ∴ base AD = base DB

                                  (i. 4)

       Wherefore *the straight line,* &c.   Q. E. F.

PROPOSITION XI. PROBLEM.

To draw a straight line at right angles to a given straight line, from a given point in the same.

Let AB be the given st. line, and C the given pt. in it. It is required to draw a st. line from C ⊥ AB.
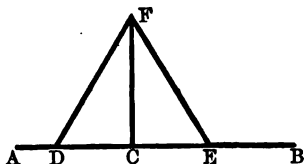
Constr. In AC take any pt. D, and make CE = CD ;
(i. 3)

On DE describe the equilat. △ DEF, (i. 1) and join CF,
Then CF drawn from the pt. C shall be ⊥ AB.

Dem. ∵ DC = CE, (constr.) and FC is common to
△s DCF, ECF ;

∴ the two sides DC, CF = the two sides EC, CF, each
to each ;

and the base DF = the base EF,   (def. 24)

∴ ∠ DCF = ∠ ECF ; (i. 8) and these two ∠s are adj.
∠s.

But when a st. line standing on another st. line makes
the adj. ∠s equal to one another, each of these ∠s is a
rt. ∠ :   (def. 10)

∴ each of ∠s DCF, ECF is a rt. ∠.

Therefore, *from the given pt. C in the given st. line AB,
CF has been drawn at right angles to AB.* Q. E. F.

Cor. By help of this problem, it may be demonstrated
that two st. lines cannot have a common segment.

If possible, let the segment AB be common to the two st. lines ABC, ABD.
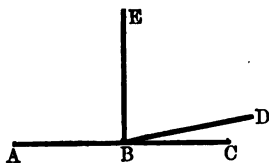
From B draw BE ⊥ AB.

Then ∵ ABC is a st. line, (hyp.) ∴ ∠ ABE = ∠ EBC ;
(def. 10)

Similarly ∵ ABD is a st. line, (hyp.) ∴ ∠ ABE = ∠ EBD ; but ∠ ABE = ∠ EBC, ∴ ∠ EBD = ∠ EBC ;
(ax. 1)

the less ∠ = the greater ∠, which is impossible.

∴ two st. lines cannot have a common segment.

[Avoid a very common improper application of prop. 8. After saying, 'and the base DF = the base EF,' boys often go on '∴ the △s DCF, FCE are equal in all respects,—by i. 8,' &c. Now Euclid does not prove this in prop. 8, and it is an error to apply that prop. in this manner. This remark might be repeated in every prop. in which prop. 8 is referred to.

The proof of the corollary is unsound, because the prop. does not tell us how to draw a perpendicular to a line *from its extremity*. It might come after prop. 13: the proof is left as an exercise for the student. It would however perhaps be better to class the theorem that 'two st. lines cannot have a common segment' with the axioms, since it seems to proceed from the definitions, and it is tacitly assumed in prop. 1; (viz. that AC, BC, cannot have a common segment at C).]

Describe a circle which shall pass through two given points and have its centre in a given straight line.

Is this always possible?

PROPOSITION XII.   PROBLEM.

To draw a straight line perpendicular to a given straight line of unlimited length, from a given point without it.

Let AB be the given st. line, which may be produced any length both ways, and let C be the given pt. without it.

It is required to draw a st. line from C ⊥ AB.
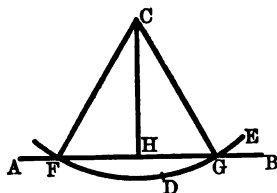
Constr. Take any pt. D on the other side of AB, and from centre C at distance CD, describe ⊙ EGF meeting AB, produced if necessary, in F and G;   (post. 3) bisect FG in H, (i. 10) and join CH, CF, CG.

Then the st. line CH shall be ⊥ the given st. line AB.
Dem. ∵ FH = HG (constr.), and HC is common to △s FHC, GHC; the two sides FH, HC = GH, HC, each to each; and base CF = base CG;   (def. 15)
∴ ∠ FHC = ∠ GHC (i. 8), and these are adj. ∠ s.

But, when a st. line standing on another st. line makes the adj. ∠ s equal to each other, each of these ∠ s is called a rt. ∠ ; and the st. line which stands on the other is called a ⊥ to it.   (def. 10)

Therefore *from the given pt. C a perpendicular CH has been drawn to the given straight line AB.*   Q. E. F.

[If the straight line were not ' of unlimited length,' it might not meet the circle.]

If the perpendicular from the vertex of a triangle on the base bisect the base, the triangle is isosceles.

PROPOSITION XIII.   THEOREM.

The angles which one straight line makes with another on one side of it, are either two right angles, or are together equal to two right angles.

Let the st. line AB make with CD, on one side of it, the ∠s CBA, ABD.

Then these are either two rt. ∠s, or are together = two rt. ∠s.
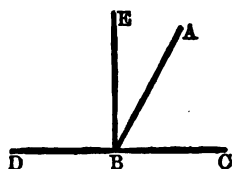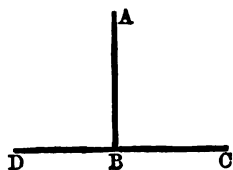
For, if ∠ CBA = ∠ DBA, each of them is a rt. ∠.
(def. 10)

Constr. But if ∠ CBA be not = ∠ DBA, from B draw
BE ⊥ CD.   (i. 11)
Dem. Then, ∠s CBE, EBD are two rt. ∠s.   (def. 10)
And ∵ ∠ CBE = ∠s CBA, ABE, add ∠ EBD to each;
∴ ∠s CBE, EBD = the three ∠s CBA, ABE, EBD.
(ax. 2)

Again ∵ ∠ DBA = the two ∠s DBE, EBA, add ∠ ABC to each;
∴ ∠s DBA, ABC = the three ∠s DBE, EBA, ABC.
(ax. 2)

But ∠s CBE, EBD have been proved = the same three ∠s.
∴ ∠s CBE, EBD = ∠s DBA, ABC;   (ax. 1)

but CBE, EBD are two rt. ∠ s; ∴ DBA, ABC are
together = two rt. ∠ s.

Therefore, *when a straight line*, &c. Q. E. D.

[The learner should find out for himself the *hypothesis*
and *conclusion* in this theorem, and in all which follow.]

DB meets the straight line ABC in B; BE, BF
bisect the angles DBC, ABD. Show that FBE is a right
angle.

PROPOSITION XIV.   THEOREM.

If at a point in a straight line, two other straight lines,
on the opposite sides of it, make the adjacent angles together
equal to two right angles ; these two straight lines shall be
in one and the same straight line.

At the pt. B in the st. line AB, let the two st. lines
BC, BD on the opposite sides of AB, make the adj. ∠ s
ABC, ABD together = two rt. ∠ s.

Then BD shall be in the same st. line with CB.
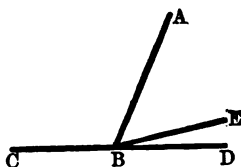
For, if BD be not in the same st. line with CB, let
BE be in the same st. line with it.

Dem. Then ∵ AB makes with the st. line CBE, on one
side of it, ∠ s CBA, ABE ; ∴ these ∠ s are together
= two rt. ∠ s ;   (i. 13)
but ∠ s CBA, ABD are together = two rt. ∠ s : (hyp.)
∴ ∠ CBA, ABE = ∠ s CBA, ABD ;   (ax. 11)
take away from each of these equals ∠ CBA, and the
rem. ∠ ABE = rem. ∠ ABD :   (ax. 3)
the less = the greater, which is impossible ; ∴ BE is
not in the same st. line with CB.

And in the same way it may be demonstrated, that
no other can be in the same st. line with it but BD,
which therefore is in the same st. line with CB.

Therefore, *if at a point,* &c.   Q. E. D.

PROPOSITION XV.   THEOREM.

If two straight lines cut one another, the vertical, or opposite angles shall be equal.

Let the two st. lines AB, CD cut one another at E.

Then shall ∠ AEC = ∠ DEB, and ∠ CEB = ∠ AED.

Dem. ∵ AE makes with CD at E ∠s CEA, AED;
   these ∠s together = two rt. ∠s.   (i. 13);

   Again ∵ DE makes with AB at E ∠s AED, DEB;
these ∠s also together = two rt. ∠s.   (i. 13)

   but ∠s CEA, AED have been shown together = two rt. ∠s;

          ∴ ∠s CEA, AED = ∠s AED, DEB;

   take away from each of these equals the common ∠ AED, and remaining ∠ CEA = remaining ∠ DEB.
                                                  (ax. 3)

   In the same way it may be demonstrated that ∠ CEB = ∠ AED.

   Therefore, *if two straight lines*, &c.   Q. E. D.

Cor. 1. From this it is manifest, that if two st. lines cut each other, the ∠s which they make at the pt. where they cut, together = four rt. ∠s.

Cor. 2. And consequently that all the ∠s made by any number of lines, meeting at one pt., together = four rt. ∠s.

[The learner should work out the corollaries. No construction is required in this theorem: no more lines are needed than those specified in the particular enunciation.]

1. Two straight lines AB, CD intersect at E ; show that the bisectors of the angles AED, BEC are in the same straight line.

2. From two given points on the same side of a given line, draw two lines which shall meet in that line and make equal angles with it.

PROPOSITION XVI. THEOREM.

If one side of a triangle be produced, the exterior angle shall be greater than either of the interior opposite angles.

Let ABC be △, and let any side BC be produced to D.

Then the ext. ∠ ACD shall be greater than either of the int. opp. ∠s CBA or BAC.
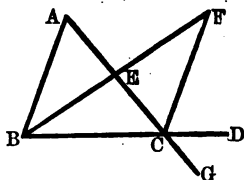
Constr. Bisect AC at E (i. 10). Join BE, and prodúce BE to F, making EF = BE, (i. 3) and join FC.

Dem. ∵ AE = EC, and BE = EF; (constr.) ∴ the two sides AE, EB = the two CE, EF, each to each in △s ABE, CFE; and ∠ AEB = ∠ CEF; (i. 15)
   ∴ △ AEB = △ CEF;   (i. 4)
and the rem. ∠s of the one = the rem. ∠s of the other, each to each, to which the = sides are opposite;
   ∴ ∠ BAE = ECF;
but ∠ ECD or ACD is greater than ∠ ECF; (ax. 9)
   ∴ ∠ ACD is greater than ∠ BAE or BAC.

In the same manner, if BC be bisected, and AC produced to G; it may be shown that ∠ BCG, that is ∠ ACD (i. 15) is greater than ∠ ABC.

Therefore, *if one side of a triangle*, &c.   Q. E. D.

[In repeating or writing out this prop., do not omit to show how ∠ ACD may be proved greater than ∠ ABC—a common mistake.]

Any two exterior angles of a triangle are together greater than two right angles.

PROPOSITION XVII. THEOREM.

Any two angles of a triangle are together less than two right angles.

Let ABC be △.

Then any two of its ∠ s together shall be less than two rt. ∠ s.
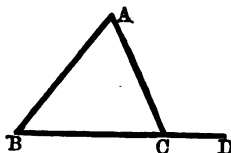
Constr. Produce any side BC to D.

Dem. Then, ∵ ACD is ext. ∠ of △ ABC, it is greater than int. opp. ∠ ABC; (i. 16)

to each of these unequals add ∠ ACB;

∴ ∠ s ACD, ACB are greater than ∠ s ABC, ACB;

(ax. 4)

but ∠ s ACD, ACB together = two rt. ∠ s; (i. 13)

∴ ∠ s ABC, ACB together are less that two rt. ∠ s.

In like manner it may be demonstrated that ∠ s BAC, ACB are together less than two rt. ∠ s, as also ∠ s CAB, ABC.

Therefore, *any two angles of a triangle*, &c. Q. E. D.

[This theorem proceeds at once as a corollary to prop. 16. The learner should make the construction and work out the prop. for the other angles.]

PROPOSITION XVIII. THEOREM.

The greater side of every triangle is opposite to the greater angle.

[That is, the greater side of every triangle has the greater angle opposite to it.]

Let ABC be △, of which AC is greater than AB.
Then shall ∠ ABC be greater than ∠ ACB.
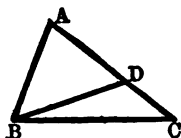
Constr. Since AC is greater than AB, make AD = AB, (i. 3) and join BD.

Dem. Then, ∵ AD = AB ∴ ∠ ADB = ∠ ABD; (i. 5)
    but ∵ the side CD of △ DBC is produced to A,
    ∴ ext. ∠ ADB is greater than int. opp. ∠ DCB;
                                                    (i. 16)
        but ∠ ADB has been proved = ∠ ABD,
            ∴ ∠ ABD is greater than ∠ DCB;
        much more is ∠ ABC greater than ∠ ACB.
    Therefore, *the greater side,* &c.   Q. E. D.

PROPOSITION XIX. THEOREM.

The greater angle of every triangle is subtended by the greater side, or, has the greater side opposite to it.

Let ABC be △ of which ∠ ABC is greater than ∠ BCA.

Then AC is greater than AB.
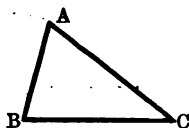
Dem. For if AC be not greater than AB, AC must either be equal to or less than AB.

     If AC = AB, then ∠ ABC = ∠ ACB;   (i. 5)
but it is not equal, (hyp.) ∴ AC is not = AB.

Again, if AC were less than AB, then ∠ ABC would be less than ∠ ACB;   (i. 18)
but it is not less, ∴, AC is not less than AB;
and AC has been shown to be not = AB,
       ∴ AC is greater than AB.

Therefore, *the greater angle*, &c.   Q. E. D.

[Beginners are apt to confuse together props. 18 and 19 : at the examination of one of our foremost public schools this year, 14 boys out of 40 in the first class wrote out the wrong prop. There is in fact a little ambiguity, and care should be taken to observe the distinction. The first, in which the hypothesis is that one side is greater than one side, is demonstrated *directly*; the second, when it is given that one angle in the one is greater than one angle in the other, is demonstrated indirectly; they thus follow, in this respect, the order of props. 5 and 6.]

PROPOSITION XX. THEOREM.

Any two sides of a triangle are together greater than the third side.

Let ABC be △.

Then any two sides shall be together greater than the third side, namely BA, AC greater than BC; AB, BC greater than AC; and BC, CA greater than AB.
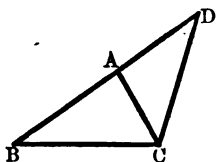
Constr. Produce any side BA to D, make AD = AC, (i. 3) and join DC.

Dem. ∵ AD = AC, (constr.) ∴ ∠ ADC = ∠ ACD;
(i. 5)
    but ∠ BCD is greater than ∠ ACD;   (ax. 9)
    ∴ also ∠ BCD is greater than ∠ ADC.

And ∵ in △ DBC, ∠ BCD is greater than ∠ BDC,
    ∴ BD is greater than BC;   (i. 19)
    but DB = BA and AC,
    ∴ BA, AC are greater than BC.

In the same way it may be demonstrated that AB, BC are greater than CA, also that BC, CA are greater than AB.

Therefore, *any two sides*, &c.   Q. E. D.

[Do not omit, before passing on, to work out the cases not demonstrated.]

1. Any three sides of a quadrilateral are together greater than the fourth.

2. The sum of the distances of any point from the angular points of a quadrilateral is greater than half the sum of the sides.

3. The sum of the diagonals of a quadrilateral is less than the sum of any four lines that can be drawn from any point whatever (except the intersection of the diagonals) to the four angles.

4. In rider 2 to prop. 15, show that the sum of the two straight lines is less than the sum of any other two lines drawn from the given points to any other point in the line.

5. How many triangles having two sides five feet and six feet long can be formed so that the third side shall contain a whole number of feet?

PROPOSITION XXI.   THEOREM.

If from the ends of a side of a triangle, there be drawn two straight lines to a point within the triangle; these shall be less than the other two sides of the triangle, but shall contain a greater angle.

Let ABC be △, and from the pts. B, C, the ends of the side BC, let the two st. lines BD, CD be drawn to a pt. D within the △.

Then BD and DC shall be less than BA and AC, the other two sides of the △, but shall contain ∠ BDC greater than ∠ BAC.
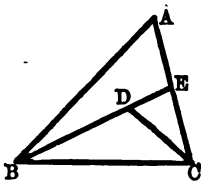
Constr. Produce BD to meet AC at E.

Dem. ∵ two sides of △ are together greater than the third side, (i. 20) ∴ BA, AE are greater than BE ;
to each of these unequals add EC ;

∴ BA, AC are greater than BE, EC.   (ax. 4)

Again ∵ CE, ED are greater than DC ;   (i. 20)
to each of these unequals add DB ;

∴ CE, EB are greater than CD, DB.   (ax. 4)

But it has been shown that BA, AC are greater than BE, EC ;

much more then are BA, AC, greater than BD, DC.

Again, ∵ ext. ∠ of △ is greater than the int. opp. ∠

(i. 16)

∴ ext. ∠ BDC of △ CDE is greater than ∠ CED;
for the same reason ext. ∠ CED of △ ABE is greater than
∠ BAC;
and it has been demonstrated, that ∠ BDC is greater
than ∠ CEB;
much more then is ∠ BDC greater than ∠ BAC.

Therefore, *if from the ends of the side,* &c.   Q.E.D.

1. Three points A, B, C are taken inside a triangle
PQR : prove that the perimeter of the triangle ABC is
less than that of the triangle PQR.

2. P is a given point, ABC a given triangle; show
that twice the sum of PA, PB, PC is greater than the
sum of the sides.

3. ABDE is a quadrilateral within the triangle ABC :
the sides of the triangle are together greater than the sides
of the quadrilateral.

PROPOSITION XXII.   PROBLEM.

To make a triangle of which the sides shall be equal to three given straight lines, but any two whatever of these must be greater than the third.

Let AP, BQ, CR be the three given st. lines, of which any two whatever are greater than the third, (i. 20) namely AP and BQ greater than CR; AP and CR greater than BQ; and BQ, CR greater than AP.

It is required to make Δ of which the sides shall be equal to AP, BQ, CR, each to each.
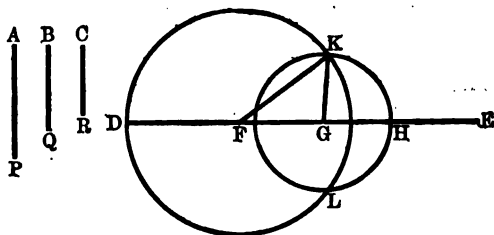
Constr. Take a st. line DE terminated at D, but unlimited towards E; make DF = AP, FG = BQ, and GH = CR;   (i. 3)

from centre F, at distance FD, describe the ⊙ DKL;
                                                         (post. 3)

and from centre G, at distance GH, describe ⊙ HLK;
and join KF, KG.

Then shall Δ KFG have its sides = the three st. lines AP, BQ, CR.

Dem. ∵ F is the centre of ⊙ DKL, ∴ FD = FK;
                                                         (def. 15)

but FD = AP, (constr.) ∴ FK = AP.

Again, ∵ G is the centre of ⊙ HLK, ∴ GH = GK ;

(def. 15)

but GH = CR, (constr.) ∴ GK = CR; and FG = BQ;

(constr.)

∴ the three st. lines KF, FG, GK, are respectively = the three AP, BQ, CR :

∴ △ *KFG has its three sides equal to the three given lines.*   Q. E. F.

['but any two whatever of these must be greater than the third.' Otherwise no triangle could be formed, as has been shown in prop. 20. The beginner may, with ruler and compasses, draw diagrams for the cases where the sum of two of the lines is equal to, and when it is less than the third line, and they will exhibit the impossibility of making a triangle.]

From a given point draw three lines of given lengths, so that their extremities may be in one line and equally distant from each other.

When is this not possible ?

PROPOSITION XXIII. PROBLEM.

At a given point in a given straight line, to make a rectilineal angle equal to a given rectilineal angle.

Let AB be the given st. line, and A the given pt. in it, and DCE the given rectil. ∠.

It is required to make an ∠ at A, in the given line AB, that shall be = the given rectil. ∠ DCE.
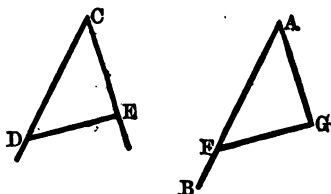
Constr. In CD, CE take any pts. D, E, and join DE; make △ AFG whose sides shall = the three st. lines CD, DE, EC, so that AF = CD, AG = CE, FG = DE.

(i. 22)

Then shall ∠ FAG = ∠ DCE.

Dem. ∵ FA, AG = DC, CE, each to each, and base
FG = base DE ;   (constr.)
∴ ∠ FAG = ∠ DCE.   (i. 8)

Wherefore, *at the given pt. A in the given st. line AB,* ∠ *FAG is made* = ∠ *DCE.*   Q. E. F.

PROPOSITION XXIV.   THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides equal to them, of the other; the base of that which has the greater angle shall be greater than the base of the other.

Let ABC, DEF be two △s, which have AB, AC = DE, DF, each to each, namely AB = DE, and AC = DF; but ∠ BAC greater than ∠ EDF.

Then shall base BC be greater than base EF.
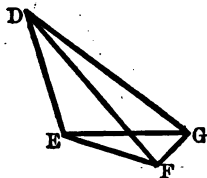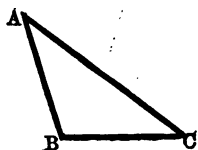
Constr. Of the two sides DE, DF, let DE be not greater than DF; at D in DE, make ∠ EDG = ∠ BAC;

(i. 23.)

make DG = DF or AC, (i. 3) and join EG, GF.

Dem. Then ∵ DE = AB, and DG = AC, (hyp.) the two sides DE, DG = AB, AC each to each, and ∠ EDG = ∠ BAC; ∴ base EG = base BC.   (i. 4)

And ∵ DG = DF ∴, ∠ DFG = ∠ DGF;   (i. 5)

but ∠ DGF is greater than ∠ EGF;   (ax. 9)

∴ ∠ DFG is greater than ∠ EGF;

∴ much more is ∠ EFG greater than ∠ EGF.

And ∵ in △ EFG, ∠ EFG is greater than ∠ EGF,

∴ EG is greater than EF ;   (i. 19)

But EG was proved $=$ BC ; $\therefore$ BC is greater that EF.
Wherefore, *if two triangles*, &c.   Q. E. D.

[Do not forget to insert the restriction 'of the two sides DE, DF, let DE be not greater than DF.' Without this there would be three cases, for F might fall on, above, or below EG. But with this condition F must fall below EG, for if DF cuts EG in H, $\angle$ DHG is greater than $\angle$ DEG,
(i. 16)
and $\angle$ DEG is not less than $\angle$ DGE, since by supposition
DF (which $=$ DG) is not less than DE,   (i. 18)
$\therefore$ $\angle$ DHG is greater than $\angle$ DGE,
$\therefore$ DH is less than DG   (i. 19)
or, is less than DF, since DF $=$ DG.]

PROPOSITION XXV.   THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of one greater than the base of the other; the angle contained by the sides of that which has the greater base, shall be greater than the angle contained by the sides, equal to them, of the other.

Let ABC, DEF be two △s which have AB, AC equal to DE, DF, each to each, namely AB = DE, and AC = DF; but base BC greater than base EF.

Then shall ∠ BAC be greater than ∠ EDF.
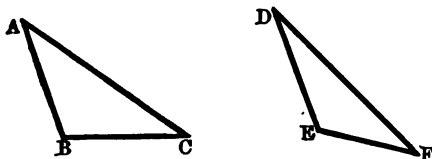
Dem. For, if ∠ BAC be not greater than ∠ EDF, it must either be equal to it, or less than it.

If ∠ BAC = ∠ EDF, then base BC = base EF;   (i. 4) but it is not equal, ∴ ∠ BAC is not = ∠ EDF.

Again, if ∠ BAC were less than ∠ EDF,
base BC would be less than base EF;   (i. 24)
but it is not less, ∴ ∠ BAC is not less than ∠ EDF; and it has been shown, that ∠ BAC is not = ∠ EDF;
∴ ∠ BAC is greater than ∠ EDF.

Therefore, *if two triangles,* &c.   Q. E. D.

PROPOSITION XXVI. THEOREM.

If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side, namely either the sides adjacent to the equal angles in each, or the sides opposite to them; then shall the other sides be equal, each to each, and also the third angle of the one equal to the third angle of the other.

Let ABC, DEF be two △s which have ∠s ABC, BCA = ∠s DEF, EFD, each to each, namely ABC to DEF and BCA to EFD; also one side equal to one side.

First let those sides be equal which are *adjacent* to the angles that are equal, namely BC to EF.

Then the other sides shall be equal, each to each, namely, AB to DE and AC to DF, and the third ∠ BAC to the third ∠ EDF.
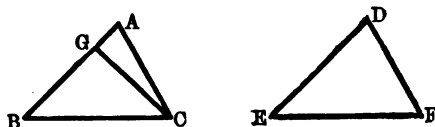
First case. Constr. For if AB be not = DE, one of them must be greater than the other; let AB be the greater. Make BG = ED, (i. 3) and join GC.

Dem. Then in the △s GBC, DEF, ∵ GB = DE, and BC = EF, (hyp.) and ∠ GBC = ∠ DEF; (hyp.)
∴ base GC = base DF, and △ GBC = △ DEF;

$$\text{(i. 4)}$$

∴ ∠ GCB = ∠ DFE;
but ∠ ACB = ∠ DFE; (hyp.)
∴ also ∠ GCB = ∠ ACB; the less equal to the greater, which is impossible;
∴ AB is not unequal to DE, that is, it is equal to it.

Hence in △s ABC, DEF, ∵ AB = DE, and BC = EF, and ∠ ABC = ∠ DEF : ∴ base AC = base DF, and third ∠ BAC = third ∠ EDF.   (i. 4)

Secondly, let sides which are opp. one of the = ∠s in each △ be equal to one another, namely, AB = DE.

In this case likewise the other sides shall be equal, AC to DF, and BC to EF, and the third ∠ BAC to third ∠ EDF.
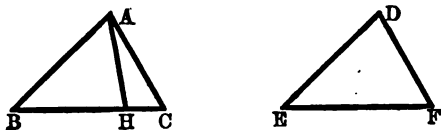
Second case.   Constr. For if BC be not = EF, one of them must be greater than the other.

Let BC be greater than EF; make BH = EF, (i. 3) and join AH.

Dem. Then in △s ABH, DEF, ∵ AB = DE, BH = EF, and ∠ ABH = ∠ DEF ;   (hyp.)

∴ base AH = base DF and △ ABH = △ DEF, (i. 4)
                    ∴ ∠ BHA = ∠ EFD ;

but ∠ EFD = ∠ BCA (hyp.); ∴ ∠ BHA = ∠ BCA,
                                        (ax. 1)

that is, the ext. ∠ of △ AHC = int. opp. ∠ BCA ; which is impossible ;   (i. 16)

∴ BC is not unequal to EF, that is BC = EF.

Hence in △s ABC, DEF; ∵ AB = DE, BC = EF, and ∠ ABC = ∠ DEF ; ∴ base AC = base DF, (i. 4)
          and third ∠ BAC = third ∠ EDG.

Therefore, *if two triangles*, &c.   Q. E. D.

[This is the third case of the equality of two △s, the two others being props. 4 and 8. Observe that the two cases (1) when the equal sides are adjacent, (2) when they are opposite the equal angles, are quite independent. It is a very common error to prove the equality of two other sides,

each to each, and then to stop with the idea that the demonstration is completed. This prop. ends the first section of the first Book.]

1. From a given point draw a line making equal angles with two given lines.

2. In a given straight line to find a point such that the perpendiculars from it upon two given straight lines are equal.

3. If two right-angled triangles have two sides containing an acute angle of the one equal to two sides containing an acute angle of the other, each to each; show that the triangles are equal in all respects.

4. If the straight line which bisects the vertical angle of a triangle also bisects the base, the triangle is isosceles.

5. In the figure of prop. 5, if FC, BG intersect in H, show that FH, GH are equal.

6. In the same figure, show that AH bisects the angle BAC.

7. The perpendicular is the shortest line that can be drawn from a given point to a given line, and that which is nearer to the perpendicular is less than one more remote; and from the same given point there can be drawn only two equal lines to the given line, one on each side of the shortest line.

PROPOSITION XXVII.   THEOREM.

If a straight line, falling on two other straight lines, make the alternate angles equal to one another, these two straight lines shall be parallel.

Let the st. line EF, which falls on the two st. lines AB, CD, make the alternate ∠s AEF, EFD equal to one another.

Then shall AB be ‖ to CD.
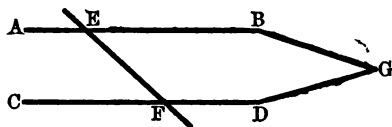
For, if not, AB and CD being produced will meet either towards A and C, or towards B and D.   Let them be produced and meet towards B and D, at the pt. G.

Dem. Then GEF is △, and its ext. ∠ AEF is greater
than the int. opp. ∠ EFG ;   (i. 16)

But ∠ AEF = ∠ EFG ;   (hyp.)

∴ ∠ AEF is greater than, and equal to, ∠ EFG, which is impossible.

∴ AB, CD being produced do not meet towards B, D.

In like manner, it may be demonstrated that they do not meet when produced towards A, C.

But those st. lines which being produced ever so far both ways do not meet, are ‖ .   (Def. 35)

∴ AB ‖ CD.

Therefore, *straight lines which*, &c.   Q. E. D.

[Alternate angles are the two angles which two straight lines make with another at its extremities, but on opposite sides of it.]

1. The quadrilateral whose diagonals bisect each other is a parallelogram.

2. A line joining two parallel straight lines is bisected: show that any other straight line drawn through the point of bisection to meet the parallel lines will be bisected in that point.

3. A line bisecting the vertical angle of a triangle meets the base in D; DE, DF parallel to the sides meet them in E, F. Prove that DE, DF are equal.

4. A line is drawn through the vertex of an isosceles triangle so as to make equal angles with the sides meeting in the vertex; prove that this line must be either parallel to the base or perpendicular to it.

PROPOSITION XXVIII. THEOREM.

If a straight line falling on two other straight lines make the exterior angle equal to the interior and opposite on the same side of the line; or make the interior angles on the same side together equal to two right angles; the two straight lines shall be parallel to one another.

Let the st. line EF, which falls on the two st. lines AB, CD, make the ext. ∠ EGB = int. and opp. ∠ GHD on the same side of EF ; or make the two int. ∠ s BGH, GHD on the same side together = 2 rt. ∠ s.

Then AB shall be ∥ CD.
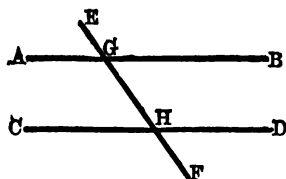
Dem. ∵ ∠ EGB = ∠ GHD, (hyp.) and ∠ EGB = ∠ AGH,
(i. 15)

∴ ∠ AGH = ∠ GHD ; (ax. i.) and they are alt. ∠ s,
∴ AB is ∥ CD. (i. 27)

Again ∵ ∠ s BGH, GHD together = two rt. ∠ s, (hyp.)
and that ∠ s AGH, BGH also together = two rt. ∠ s ;
(i. 13)

∴ ∠ s AGH, BGH = ∠ s BGH, GHD ;

take away the common ∠ BGH, and the rem. ∠ AGH
= rem. ∠ GHD, (ax. 3)
and they are alt. ∠ s ; ∴ AB ∥ CD. (i. 27)

Therefore, *if a straight line,* &c.   Q. E. D.

1. If the opposite angles of a quadrilateral are equal to one another, the figure is a parallelogram.

2. From any point D on the base of an isosceles triangle, perpendiculars are drawn to the other two (the equal) sides: prove that the sum of the perpendiculars is the same whatever be the position of D.

PROPOSITION XXIX.   THEOREM.

If a straight line fall on two parallel straight lines, it makes the alternate angles equal to one another, and the exterior angle equal to the interior and opposite angle on the same side; and also the two interior angles on the same side together equal to two right angles.

Let the st. line EF, fall on the ‖ st. lines AB, CD.

Then the alt. ∠s AGH, GHD shall be = one another; the ext. ∠ EGB shall be = the int. and opp. ∠ GHD on the same side of the line EF ;

and the two int. ∠s BGH, GHD on the same side of
    EF shall together be = two rt. ∠ s.
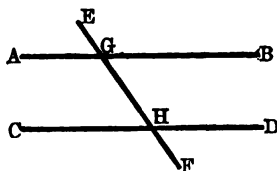
Dem. For if ∠ AGH be not = alt. ∠ GHD, one of them
    must be greater than the other ;

If possible, let AGH be greater than GHD; add to each
    of these unequals ∠ BGH ;

∴ ∠s AGH, BGH are greater than ∠s BGH, GHD ;
                                            (ax. 4)

But ∠s AGH, BGH = two rt. ∠s (i. 13); ∴ ∠s BGH,
    GHD are less than two rt. ∠ s.

But if a st. line meets two st. lines, so as to make the
    two int. ∠s on the same side of it, taken together, less
    than two rt. ∠s ; these st. lines, being continually pro-
    duced, shall at length meet on that side on which are
    the ∠s which are less than two rt. ∠ s.   (ax. 12)

∴ AB, CD, continually produced, will meet towards B, D;
but they never meet, since by hypothesis they are ∥ ;
∴ ∠ AGH is not unequal to ∠ GHD, that is
∠ AGH = alt. ∠ GHD.
But ∠ AGH = ∠ EGB; (i. 15) ∴ ∠ EGB = ∠ GHD:

(ax. i.)

add to each ∠ BGH ;

∴ ∠ s EGB, BGH = ∠ s BGH, GHD ;   (ax. 2)

but EGB, BGH = two rt. ∠ s ; (i. 13) ∴ also

BGH, GHD = two rt. ∠ s.   (ax. 1)

Therefore, *if a st. line,* &c.   Q. E. D.

[This is the converse of props. 27 and 28.   There is some
difficulty in the theory of parallel lines, because as the defi-
nition is of a negative character (that they never meet), it is
necessary to assume some positive quality on which reason-
ings may be founded, and this Euclid has done in ax. 12,
which he regards as a self-evident theorem.   Thirty methods
have been proposed as starting points (for a starting point
there must be), but on the whole Euclid's axiom is the
simplest and least liable to objection.]

If the straight line bisecting the exterior angle of a
triangle be parallel to the base, the triangle is isosceles.

PROPOSITION XXX.   THEOREM.

Straight lines which are parallel to the same straight line are parallel to one another.

Let AB, CD be each of them ‖ EF.
Then also AB shall be ‖ CD.
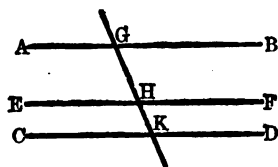
Constr. Let st. line GHK cut AB, EF, CD.
Dem. Then ∵ GHK cuts ‖ st. lines AB, EF at G, H ;
$\qquad\qquad$ ∴ ∠ AGH = alt. ∠ GHF. $\qquad$ (i. 29)
$\quad$ Again ∵ GHK cuts ‖ st. lines EF, CD at H, K ;
$\qquad\qquad$ ∴ ext. ∠ GHF = int. ∠ HKD ; $\qquad$ (i. 29)
$\qquad$ and it was shown that ∠ AGH = ∠ GHF ;
$\qquad\qquad$ ∴ ∠ AGH = ∠ GKD ;
$\qquad$ and these are alt. ∠ s, ∴ AB is ‖ CD. $\quad$ (i. 27)
$\quad$ Therefore, *straight lines which are parallel,* &c.

$\qquad\qquad\qquad\qquad\qquad\qquad$ Q. E. D.

PROPOSITION XXXI. PROBLEM.

To draw a straight line through a given point parallel to a given straight line.

Let A be the given pt., BC the given st. line.

It is required to draw through the pt. A a st. line ∥ BC.
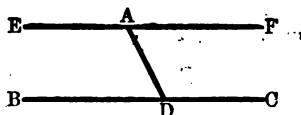
Constr. In BC take any pt. D, join AD, and at the pt. A
make ∠ DAE = ∠ ADC ;   (i. 23)

and produce EA to F.   Then shall EF be ∥ BC.

Dem. ∵ AD, which meets EF, BC, makes alt. ∠s
EAD, ADC equal to one another,   (constr.)

∴ EF is ∥ BC.   (i. 27)

Therefore, *through the given pt. A, has been drawn a
st. line EAF ∥ the given st. line BC.*   Q. E. F.

1. Given a point, an angle, and a straight line. From
the point draw a line to the given line making with it an
angle equal to the given angle.

2. Draw a line DE parallel to the base BC of a
triangle ABC, so that DE shall be equal to the sum of
BD, CE.

3. Draw a line DE parallel to the base BC of a
triangle ABC, so that DE shall be equal to the difference
of BD, CE.

4. Through a given point draw a line so that the part
of it intercepted between two given parallel lines may be
equal to a given straight line. Show that there may be
two, one, or no solutions.

5. Through two given points draw two lines forming
with a line given in position an equilateral triangle.

PROPOSITION XXXII. THEOREM.

If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are equal to two right angles.

Let ABC be a △, and let one of its sides BC be produced to D. Then the ext. ∠ ACD shall = the two int. and opposite ∠s CAB, ABC: and the three int. ∠s ABC, BCA, CAB shall together = two rt. ∠s.
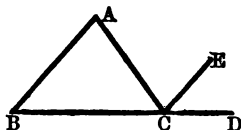
Constr. Through C draw CE ∥ BA.   (i. 31)
Dem. Then ∵ CE is ∥ BA and AC meets them, ∴
  ∠ ACE = alt. ∠ BAC.   (i. 29)
Again ∵ CE ∥ AB, and BD falls on them, ∴ ext.
  ∠ ECD = int. and opp. ∠ ABC ;   (i. 29)
        but ∠ ACE was shown = ∠ BAC ;
∴ whole ext. ∠ ACD = two int. and opp. ∠s CAB, ABC :
                                                 (ax. 2)
      to each of these equals add ∠ ACB,
∴ ∠s ACD and ACB = the three ∠s CAB, ABC, ACB;
                                                 (ax. 2)
    but ∠s ACD, ACB together = two rt. ∠s, (i. 13)
∴ ∠s CAB, ABC, ACB together = two rt. ∠s. (ax. 1)
    Therefore, *if a side of any triangle,* &c.   Q. E. D.
Cor. 1. *All the interior angles of any rectilineal figure to-
    gether with four right angles, are equal to twice as many
    right angles as the figure has sides.*

For any figure ABCDE can be divided into as many △s as it has sides, by drawing st. lines from a pt. F within it to each ∠.
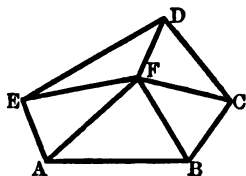
Now the three int. ∠s of each of these △s = two rt. ∠s. ∴ all the ∠s of all these △s = twice as many rt. ∠s as the figure has sides. *

But the same ∠s of these △s = all the int. ∠s of the figure together with ∠s at F; and the ∠s at F = four rt. ∠s, (i. 15. Cor. 2) ∴ the same ∠s of these △s = all the int. ∠s of the figure together with four rt. ∠s.

But it has been proved (*) that the ∠s of the △s are = to twice as many rt. ∠s as the figure has sides ; ∴ all the int. ∠s of the figure together with four rt. ∠s = twice as many rt. ∠s as the figure has sides.

Cor. 2. All the exterior angles of any rectilineal figure, made by producing the sides successively in the same direction, are together equal to four right angles.
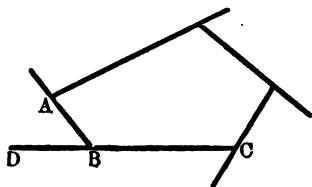
Since every int. ∠ such as ABC, with its adj. ext. ∠ABD = two rt. ∠s, (i. 13)

∴ all the int. ∠ s, together with all the ext. ∠ s, = twice as many rt. ∠ s as the figure has sides ; but by Cor. 1, all the int. ∠ s together with four rt. ∠ s = twice as many rt. ∠ s as the figure has sides ;

∴ all the int. ∠ s together with all the ext. ∠ s are equal to all the int. ∠ s and four rt. ∠ s ;

take away from these equals all the int. ∠ s,

∴ all the ext. ∠ s of the figure are equal to four rt. ∠ s.

[From this prop. it follows : (1) that each of the equal angles of an isosceles right-angled triangle is half a right angle ; (2) that each of the three angles of an equilateral triangle is two-thirds of a right angle ; (3) when two angles of a triangle are together equal to the third, the latter is a right angle.]

1. On a given line describe a square of which the given line shall be a diagonal.

2. The difference of the angles at the base of any triangle is double the angle contained by two lines drawn from the vertex, one bisecting the vertical angle, the other perpendicular to the base.

3. Straight lines bisecting two adjacent angles of a parallelogram are at right angles to one another.

4. If the interior angle at one angular point of a triangle and the exterior angle at another be bisected by straight lines, the angle contained by the two bisecting lines is equal to half the third angle of the triangle.

5. ABC is a triangle right-angled at A with the angle B double of the angle C ; show that CB is double of AB.

6. In the figure to i. 5, if BG, FC intersect at H, and if the angle FBG is equal to the angle ABC, then the angle BHF is equal to twice the angle BAC.

7. To construct a triangle when one angle, a side opposite to it, and the sum of the other two sides are given.

8. What figure is that, whose exterior angles (formed by producing the sides successively) are together equal to its interior angles.

9. How many sides has a figure, if the sum of its interior is double that of its exterior angles?

10. ABC is an equilateral triangle; D, E are points in BC, CA respectively such that BD is equal to CE; if AD, BE be joined and intersect in O, show that the angle AOB is twice an angle of the equilateral triangle.

11. If each of the equal angles of an isosceles triangle be one-fourth the vertical angle, and from one of them a perpendicular is drawn to the base meeting the opposite side produced; then will the part produced, the perpendicular, and the remaining side form an equilateral triangle.

PROPOSITION XXXIII.   THEOREM.

The straight lines which join the extremities of two equal and parallel straight lines towards the same parts, are themselves equal and parallel.

Let AB, CD be = and ∥ st. lines, joined towards the same parts by AC, BD.

Then AC, BD shall be = and ∥ .
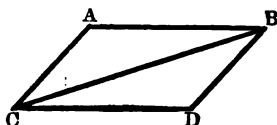
Constr.   Join BC.

Dem. Then, ∵ AB is ∥ CD, (hyp.) and BC meets them,

  ∴ ∠ABC = alt. ∠BCD ;   (i. 29)

and ∵ AB = CD, (hyp.) and BC is common to the two

  △s ABC, DCB, and ∠ABC = ∠BCD ;   ∴ base

  AC = base BD,   (i. 4)

and △ ABC = △ BCD, and the other ∠s = the other ∠s,

  each to each, to which the = sides are opp. ;

            ∴ ∠ACB = ∠CBD.

  And ∵ BC meets AC, BD, and makes the alt. ∠s

          ACB, CBD equal to one another ;

  ∴ AC is ∥ BD ; (i. 27) and AC was shown = BD.

  Therefore, *straight lines which,* &c.   Q. E. D

PROPOSITION XXXIV. THEOREM.

The opposite sides and angles of parallelograms are equal to one another, and the diameter bisects them, that is, divides them into two equal parts.

Let ACDB be a ▱ of which BC is a diameter.
. · Then must AB = CD ; AC = BD ; and △ ABC = △ BCD.
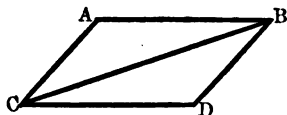
Dem. ∵ AB is ∥ CD, and BC meets them,

∴ ∠ ABC = ∠ BCD. (i. 29)

and ∵ AC is ∥ BD, and BC meets them,

∴ ∠ ACB = ∠ CBD. (i. 29)

Hence in △s ABC, CBD, ∵ the two ∠s ABC, BCA in the one, = BCD, CBD in the other, each to each; and one side BC, which is adj. to their = ∠s is common ;

∴ their other sides are =, each to each, and the third ∠ of the one = the third ∠ of the other ; (i. 26)

namely AB=CD, AC=BD, and ∠ BAC=∠ BDC.

And ∵ ∠ ABC=∠ BCD and ∠ CBD=∠ ACB,

∴ whole ∠ ABD = whole ∠ ACD ; (ax. 2)

and ∠ BAC has been shown = ∠ BDC.

∴ *the opp. sides and ∠s of a ▱ are equal to one another.*

Also, the diameter bisects it.

For since AB=CD, BC is common, and ∠ ABC has
been shown = ∠ BCD ;

∴ △ ABC = △ BCD ; (i. 4) and *the diameter bisects it.*

Q. E. D.

[Avoid two common errors : (1) when applying prop. 26
do not say 'therefore the triangles are equal in all respects ;'
(2) do not stop short after proving that the opposite sides
and angles are equal. It is necessary to complete the proof
by applying prop. 4. This prop. is the last in the second
section of the first Book, which treats of the theory of parallel
lines.]

1. In what case will the diagonal bisect the angle of a
parallelogram ?

2. Straight lines which bisect two opposite angles of
a parallelogram are parallel or coincident.

3. Draw lines through the angular points of a
parallelogram which shall form another parallelogram
double of the former.

4. The sides AB, AC of a triangle are bisected in
D, E respectively, and BE, CD are produced till EF = EB,
and GD=DC ; show that the line GF passes through A.

5. From the angles C, B of a parallelogram ABCD,
CE, BE are drawn parallel to the diagonals, which EA, ED
intersect in H, K.   Show that HK is half BC.

6. The diagonals of a rhombus bisect one another at
right angles.

7. Through a given point between two straight lines
which are not parallel draw a straight line which shall be
bisected in that point.

8. Draw a line from a given point in a side of a
parallelogram which shall bisect the parallelogram.

9. Bisect a parallelogram by a line drawn at right
angles to one of the sides.

10. The diagonals AC, BD of a parallelogram inter-

sect in O, and the parallelograms AOBP, DOCQ are formed. Show that POQ is a straight line.

11. The diagonals of a parallelogram are equal : find its angles.

12. Show that a given straight line may be thus trisected; on it as diagonal describe any parallelogram : draw lines from the middle points of two opposite sides to two opposite angles : these lines trisect the given line.

PROPOSITION XXXV. THEOREM.

Parallelograms on the same base, and between the same parallels, are equal to one another.

Let the ⏥s ABCD, EBCF be on the same base BC, and between the same ∥s AF, BC.

Then shall ⏥ ABCD ⸗ ⏥ EBCF.
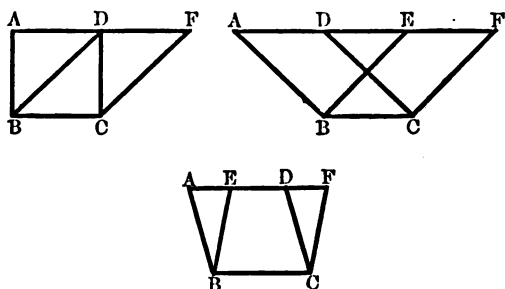
Dem. If the sides AD, DF of ⏥s ABCD, DBCF, opposite BC, be terminated at the same pt. D ;

then, since each ⏥ is double △ DBC;   (i. 34)

∴ ⏥ ABCD = ⏥ DBCF.   (ax. 6)

But if AD, EF be not terminated at the same pt. ;

Then, ∵ ABCD is ⏥ ∴ AD = BC ;   (i. 34)

for a similar reason EF=BC ; ∴ AD=EF ;   (ax. 1)

and DE is common ; ∴ the whole (or the remainder)
    AE=the whole (or the remainder) DF, (ax. 2 or 3)

∴ in △s EAB, FDC, FD = AE, DC = AB,   (i. 34)

and ext. ∠ FDC = int. ∠ EAD ;   (i. 29)

∴ △ FDC ⇌ △ EAB.   (i. 4)

From the trapezium ABCF take △ FDC, and from the same trapezium take △ EAB, and the remainders are equal, (ax. 3)

that is, ▱ ABCD = ▱ EBCF.

Therefore, *parallelograms on the same,* &c.

Q. E. D.

[In the latter part, ax. 3 is used in rather a peculiar manner. It will however be clear to the beginner that if we cut off a certain portion from a carpet and an *equal* portion from an *equal* carpet that the portions which remain are equal. Note carefully that the parallelograms are *equal in area*, but are not equal in all respects, nor identical in figure. The area of a four-sided field may be equal to that of a triangular field, and the fields may be said to be equal, but their figures are not equal. The word *equivalent* would better express equality of area, leaving the word *equal* to express *equality of area and of figure also*. Thus the parallelograms in this prop. are equivalent: the triangles in prop. 4 are equal. But the fact is there are few cases in which any ambiguity can arise, and a little care is all that is wanted.]

Describe a parallelogram equal to a given square, and having an angle equal to half a right angle.

PROPOSITION XXXVI.   THEOREM.

Parallelograms on equal bases, and between the same parallels, are equal to one another.

Let ABCD, EFGH be ⌷s on = bases BC, FG, and between the same ∥s AH, BG.

Then shall ⌷ ABCD = ⌷ EFGH.
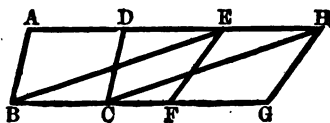
Constr. Join BE, CH.

Dem. Then ∵ BC = FG (hyp.), and FG = EH,   (i. 34)

$$\therefore BC = EH;   \text{(ax. 1)}$$

and these lines are ∥, (hyp.) and joined towards the same parts by BE, CH;

$$\therefore BE, CH \text{ are both} = \text{and} \parallel;   \text{(i. 33)}$$

∴ EBCH is a ⌷, (def. 36) and it

= ⌷ ABCD,   on the same base and between the same ∥s;   (i. 35)

For the same reason, ⌷ EFGH = ⌷ EBCH;

$$\therefore ⌷ ABCD = ⌷ EFGH.   \text{(ax. 1)}$$

Therefore, *parallelograms on equal*, &c.   Q. E. D.

PROPOSITION XXXVII. THEOREM.

Triangles on the same base, and between the same parallels, are equal to one another.

Let the △s ABC, DBC be on the same base BC, and between the same ∥s AD, BC.

Then shall △ ABC = △ DBC.
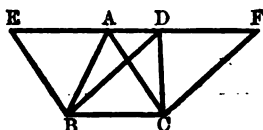
Constr. Produce AD both ways to E, F; through B draw BE ∥ CA, and through C draw CF ∥ BD.

Dem. Then each of the figures EBCA, DBCF is ▱;

and EBCA = DBCF, ∵ they are on the same base BC, and between the same ∥s BC, EF.   (i. 35)

And ∵ the diameter AB bisects ▱ EBCA;   (i. 34)

∴ △ ABC is half ▱ EBCA;

also ∵ the diameter DC bisects ▱ DBCF,   (i. 34)

∴ DBC is half ▱ DBCF,

but the halves of equal things are equal;   (ax. 7)

∴ △ ABC = △ DBC.

Therefore, *triangles*, &c.   Q.E.D.

1. Bisect a triangle by a line drawn through a given point in one of the sides.

2. Given an isosceles triangle and a point in one of its equal sides, draw a line from the point to the opposite side produced, which shall make with these sides a triangle equal to the given triangle.

PROPOSITION XXXVIII. THEOREM.

Triangles on equal bases, and between the same parallels, are equal to one another.

Let △s ABC, DEF be on = bases BC, EF, and between the same ‖s BF, AD.

Then shall △ ABC = △ DEF.
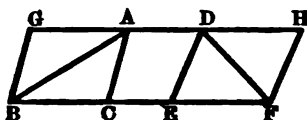
Constr. Produce AD both ways to G, H;

through B draw BG ‖ CA; and through F draw FH ‖ ED.   (i. 31)

Dem. Then each of the figures GBCA, DEFH is ▱;
(def.)

and they are equal, ∵ they are on equal bases BC, EF, and between the same ‖s BF, GH.   (i. 36)

And ∵ the diameter AB bisects ▱ GBCA,   (i. 34)
∴ △ ABC is half ▱ GBCA;

also, ∵ the diameter DF bisects ▱ DEFH,   (i. 34)
∴ △ DEF is half ▱ DEFH;

but the halves of equal things are equal;   (ax. 7)
∴ △ ABC = △ DEF.

Therefore, *triangles*, &c.   Q. E. D.

[The bases of the triangles are of course in the same straight line.]

ABC, ABD are two equal triangles on the same base and on opposite sides of it. If CD meets AB in E, then CE, ED are also equal.

PROPOSITION XXXIX. THEOREM.

Equal triangles on the same base and on the same side of it are between the same parallels.

Let the equal △s ABC, DBC be on the same base BC, and on the same side of it.

Then the △s ABC, DBC shall be between the same ∥s.
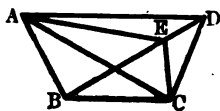
Constr. Join AD; AD shall be ∥ BC.

For, if it is not, through A draw AE ∥ BC,
(i. 31)

meeting BD or BD produced in E, and join EC.

Dem. Then △ ABC = △ EBC, ∵ they are on the same base, and between the same ∥s BC, AE; (i. 37)

but △ ABC = △ DBC; (hyp.) ∴ also △ DBC = △ EBC

the greater = the less, which is impossible :

therefore AE is not parallel to BC.

In the same way it can be shown, that no other line through A but AD is ∥ BC;

∴ AD is ∥ BC.

Therefore, *equal triangles*, &c. Q. E. D.

PROPOSITION XL.   THEOREM.

Equal triangles, on equal bases in the same straight line,
and on the same side of it, are between the same parallels.

Let the = △s ABC, DEF be on = bases BC, EF,
in the same straight line BF, and on the same side
of it.

Then they shall be between the same ∥s.
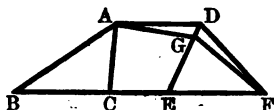
Constr. Join AD; AD shall be ∥ BF.

For, if it is not, through A draw AG ∥ BF,   (i. 31)
meeting ED at G, and join GF.

Dem. Then △ ABC = △ GEF, ∵ they are on = bases
BC, EF, and between the same ∥s BF, AG ;   (i. 38)
but △ ABC = △ DEF ; (hyp.) ∴ also △ DEF = △ GEF,
(ax. 1)

the greater = the less, which is impossible.
∴ AG is not ∥ BF.

And in the same manner it can be demonstrated
that no other st. line through A but AD is ∥ BF ;
∴ AD is ∥ BF.

Therefore, *equal triangles on*, &c.   Q. E. D.

EXAMPLES ON PROPOSITIONS 39 AND 40.

1. A quadrilateral is divided by its diagonals into four triangles of which two opposite ones are equal. Show that two sides of the quadrilateral are parallel.

2. The straight line which joins the middle points of two sides of a triangle is parallel to the base.

3. The straight line which joins the middle points of two sides of a triangle is equal to half the base.

4. Given the middle points of the sides of a triangle, to construct it.

5. If the points of bisection of the sides of a triangle be joined, the triangle so formed will be one fourth of the original triangle.

6. Two straight lines, drawn to bisect the opposite sides of any quadrilateral, will also bisect one another.

PROPOSITION XLI.   THEOREM.

If a parallelogram and a triangle be on the same base, and between the same parallels, the parallelogram is double of the triangle.

Let the ▱ ABCD and the △ EBC be on the same base BC and between the same ∥s BC, AE.

Then shall ▱ ABCD be double of △ EBD.
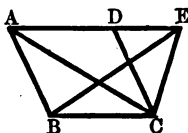
Constr. Join AC.

Dem. Then △ ABC = △ EBC, ∵ they are on the same
base BC, and between the same ∥s BC, AE.   (i. 37)

But ▱ ABCD is double of △ ABC, ∵ AC bisects
the ▱ ;   (i. 34)

∴ ABCD is also double of EBC.

Therefore, *if a parallelogram,* &c.   Q. E. D.

1. O is any point within a parallelogram ABCD, and
lines are drawn from it to the angles. Show that the
triangles OAB, OCD are together equal to the triangle
ABC.

2. If a parallelogram is double of a triangle, and they
have the same base, or equal bases on the same straight
line, and towards the same parts, they shall be between
the same parallels.

PROPOSITION XLII. PROBLEM.

To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given angle.

Let ABC be the given △ and D the given ∠.

It is required to describe a ▱ = the given △, having one of its ∠s = D.
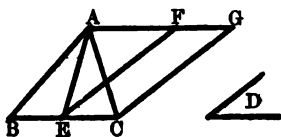
Constr. Bisect BC in E, (i. 10) and join AE;

at E make ∠ CEF = ∠ D; (i. 23)

through A draw AFG ∥ BC, and through C draw CG ∥ EF; (i. 31)

Dem. Then CEFG is a ▱. (def.)

And ∵ △s ABE, AEC are on = bases BE, CE, and between the same ∥s BC, AG, they are equal;

(i. 38)

∴ △ ABC is double of △ AEC.

But ▱ FECG is double of △ AEC, ∵ they are on the same base and between the same ∥s; (i. 41)

∴ ▱ FECG = △ ABC, (ax. 6)

and has one of its ∠s CEF = the given ∠ D.

(constr.)

Therefore, *a parallelogram has been described as required.* Q. E. F.

Describe a parallelogram equal to a given equilateral triangle having an angle equal to an angle of the triangle. Show that the perimeters are likewise equal.

PROPOSITION XLIII. THEOREM.

*The complements of the parallelograms which are about
the diameter of any parallelogram are equal to one another.*

Let ABCD be a ▱, of which AC is a diameter, and
EH, GF the ▱s *about* AC (that is, through which AC
passes); and BK, KD the other ▱s which make up the
whole figure, *and which are therefore called the comple-
ments.*

Then shall complement BK = complement KD.
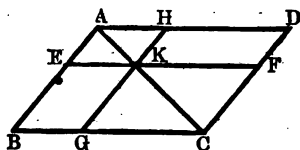
Dem. ∵ ABCD is a ▱ and AC its diameter, ∴ △ ABC
= △ ADC.   (i. 34)

Again, ∵ EKHA is a ▱ and AK its diameter, ∴ △
AEK = △ AHK;   (i. 34)

and for the same reason △ KGC = △ KFC;

∴ the two △s AEK, KGC = the two △s AHK, KFC;
                                                 (ax. 2)

but the whole △ ABC = the whole △ ADC;

∴ the remainder, the complement BK = the remainder,
the complement KD.   (ax. 3.)

Therefore, *the complements,* &c.   Q. E. D.

PROPOSITION XLIV. PROBLEM.

To a given straight line to apply a parallelogram which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let AB be the given st. line, C the given $\triangle$, and D the given $\angle$.

It is required to apply to AB a $\square = \triangle$ C, and having one of its $\angle$s $=$ D.
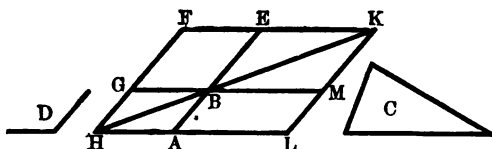
Constr. Describe a $\square$ BEFG $= \triangle$ C, having $\angle$ EBG $= \angle$ D, (i. 42)

      so that BE be in the same st. line with AB;

produce FG to H, draw AH $\parallel$ BG or EF, (i. 31) and join HB.

Dem. Then, $\because$ HF falls on the $\parallel$s AH, EF, $\therefore$ $\angle$s AHF, HFE are together $=$ two rt. $\angle$s; (i. 29)

      $\therefore$ $\angle$s BHF, HFE are less than two rt. $\angle$s:

but st. lines which with another st. line make the two int. $\angle$s on the same side less than two rt. $\angle$s, will meet on that side, if produced far enough :   (ax. 12)

      $\therefore$ HB, FE shall meet, if produced ;

let them meet in K, through K draw KL $\parallel$ EA or FH, and produce HA, GB to meet KL at L, M.

Then KLHF is a ▱, of which the diameter is HK ; ⌉
and AG, ME are the ▱s about HK, also
LB, BF are the complements ;
∴ compl. LB = compl. BF ;   (i. 43)
but BF = △ C ; (constr.) ∴ LB = △ C.
And ∵ ∠ GBE = ∠ ABM, (i. 15) and also = ∠ D ;.
                                   (constr.)
∴ ∠ ABM = ∠ D.
∴ *to the st. line AB the* ▱ *LB is applied as required.*
                                   Q. E. F.

PROPOSITION XLV. PROBLEM.

To describe a parallelogram equal to a given rectilineal figure, having an angle equal to a given rectilineal angle.

Let ABCD be the given rectil. figure, and E the given ∠.

It is required to describe a ▱ = to ABCD, having an ∠ = ∠ E.
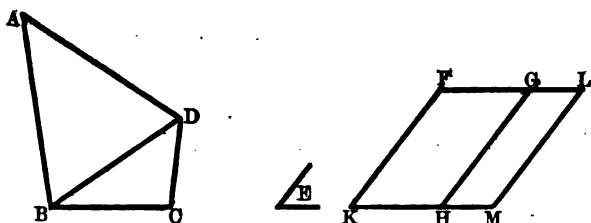
Constr. Join DB, and describe ▱ FH = △ ADB and having ∠ FKH = ∠ E;   (i. 42)

to GH apply the ▱ GM = △ DBC, having ∠ GHM = ∠ E.   (i. 44)

Then FKML shall be the ▱ required.

Dem. ∵ each of the ∠s FKH, GHM = ∠ E, (constr.)

∴ ∠ FKH = ∠ GHM : add to each of these =s ∠ KHG ;

∴ ∠s FKH, KHG = ∠s KHG, GHM,   (ax. 2)

but FKH, KHG together = two rt. ∠s;   (i. 29)

∴ KHG, GHM together = two rt. ∠s ;

and ∵ at H in GH the two st. lines KH, HM, on the opposite sides of it, make the adj. ∠s KHG, GHM = two rt. ∠s, ∴ HK is in the same st. line with HM.

(i. 14)

And ∵ HG meets the ∥s KM, FG, ∴ ∠ MHG
= alt. ∠ HGF;   (i. 29)
add to each of these =s the ∠ HGL;
∴ ∠s MHG, HGL = ∠s HGF, HGL;   (ax. 2)
but ∠s MHG, HGL = two rt. ∠s;   (i. 29)
∴ also HGF, HGL = two rt. ∠s, and
∴ FG is in the same st. line with GL.   (i. 14)
And ∵ KF is ∥ HG, and HG is ∥ ML, ∴ KF is ∥
ML :   (i. 30)
and FL has been proved ∥ KM, ∴ FKML is a ▱ ;
                                        (def.)
and since ▱ HF = △ ABD, and ▱ GM = △ DBC :
∴ the whole ▱ KFLM = the whole rectil. figure
ABCD.

Therefore, *the ▱ KFLM has been described as re-
quired.*   Q. E. F.

Cor. From this it is manifest how *to a given straight line
to apply a parallelogram* which shall be equal to a
given rectilineal figure, and shall have an angle equal
to a given rectilineal angle; namely by applying to
the given line a parallelogram equal to the first tri-
angle (i. 44), and having an angle equal to the given
angle ; and so on.

[This appears to be a difficult prop. for beginners, as some
step in the demonstration is often omitted.   The main point
is to show clearly that FGL, KHM are straight lines by
means of props. 14 and 29.   When the figure F M has thus
been proved to be a parallelogram, the rest easily follows.]

PROPOSITION XLVI. PROBLEM.

To describe a square on a given straight line.

Let AB be the given st. line.

It is required to describe a sq. on AB.
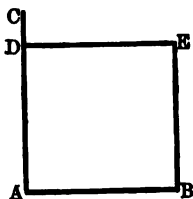
Constr. From A draw AC ⊥ AB; (i. 11) make AD = AB,   (i. 3)

through D draw DE ∥ AB, and through B, BE ∥ AD;   (i. 31)

Then shall ADEB be a sq.

Dem. Since AE is a ▱, (constr.) ∴ AB = DE and AD = BE;   (i. 34)

but AD = AB,   (constr.)

∴ AB, BE, ED, DA are all =, and the figure AE is equilateral.

Again, since AD meets the ∥s AB, DE ∴ ∠s BAD, ADE are together = two rt. ∠s;   (i. 29)

but BAD is a rt. ∠ ; (constr.) ∴ also ADE is a rt. ∠.

Now the opp. ∠s of ▱s are = ; (i. 34) ∴ each of the opp. ∠s ABE, BED is a rt. ∠ ;

∴ AE is rectangular, and it has been proved equilateral;

∴ *it is a sq.* (def. 30) *and it is described on the given st. line AB.*   Q. E. F.

Cor. Hence every ▱ that has one rt. ∠ has all its ∠s rt. ∠s.

PROPOSITION XLVII.   THEOREM.

In any right-angled triangle, the square which is described on the side subtending the right angle is equal to the squares described on the sides which contain the right angle.

Let ABC be a rt.-angled △, having the rt. ∠ BAC. Then the sq. on BC = the sqs. on BA, AC.
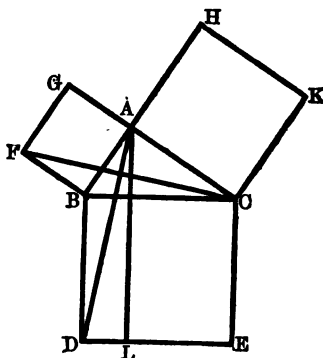
Constr. On BC describe the sq. BDEC, and on BA, AC the sqs. GB, HC;   (i. 46)
through A draw AL ∥ BD or CE; (i. 31) and join AD, FC.

Dem. ∵ ∠ BAC is a rt. ∠ (hyp.), and ∠ BAG is a rt. ∠,   (def. 30)
the two st. lines CA, AG on the opp. sides of AB, make with it at A the adj. ∠s = two rt. ∠s; ∴ CA is in the same st. line with AG.   (i. 14)

For the same reason, BA and AH are in the same st. line.

Now the rt. ∠ DBC = the rt. ∠ FBA;　　(ax. 11)

　　Add to each of these =s ∠ ABC;

∴ the whole ∠ ABD = the whole ∠ FBC.　(ax. 2)

And ∵ AB, BD = FB, BC, each to each, and the

　　included ∠ ABD = the included ∠ FBC,

　　　　∴ △ ABD = △ FBC.　　　　(i. 4)

Now ▭ BL is double of △ ABD, ∵ they are on the

same base BD and between the same ∥s BD, AL;

　　　　　　　　　　　　　　　　(i. 41)

also the sq. GB is double of △ FBC, ∵ they are on the

same base FB and between the same ∥s FB, GC.

　　　　　　　　　　　　　　　　(i. 41)

But the doubles of equals are equal to one another;

　　　　　　　　　　　　　　　　(ax. 6)

　　　　∴ ▭ BL = sq. GB.

Similarly, by joining AE, BK, it can be proved that

▭ CL = sq. HC.

∴ the whole sq. BDEC = the two sqs. GB, HC;

　　　　　　　　　　　　　　　　(ax. 2)

　　that is, the sq. on BC = the sqs. on AB, AC.

　　Therefore, *in any rt.-angled* △, &c ,　　Q. E. D.

[It is essential (as in prop. 45) to show that CAG, BAH
are st. lines. This is often omitted. The learner should
apply this prop. to find a sq. equal to the sum of any number
of given sqs., or equal to the difference of two given sqs.]

　　1. Four times the square on the perpendicular from
an angle of an equilateral triangle on the opposite side is
three times the square on one of the sides.

　　2. If A be the vertex of an isosceles triangle ABC,
and CD be perpendicular to AB, the squares on the
three sides are together equal to the square on BD, twice
the square on AD, and thrice the square on CD.

　　3. From any point in the diameter of a semicircle two
lines are drawn to the circumference, one to the middle

point of the arc, the other at right angles to the diameter. The squares on these lines are together double the square on the radius.

4. From the middle point of a side of a right-angled triangle a perpendicular is drawn to the hypotenuse : show that the difference of the squares on the segments into which it is divided is equal to the square on the other side.

5. AB is the common hypotenuse of two right-angled triangles, ACB, ADB, and AE, BF are perpendicular to CD (produced both ways if necessary). Show that the sum of the squares on CE, CF is equal to the sum of the squares on DE, DF.

6. ABCD is a rectangle, E any point. Show that the squares on EA, EC are together equal to the squares on EB, ED.

7. If two exterior angles of a triangle be bisected, the line drawn from the point of intersection of the bisecting lines to the opposite angle of the triangle will bisect that angle.

8. Find a point in the diagonal of a square produced, from which if a straight line be drawn parallel to one of the sides, and meeting another side produced, it will form with the produced diagonal and produced side a triangle equal to the square.

9. Divide a given line into two parts such that the sum of their squares may be equal to a given square. When is this impossible?

10. On AC the hypotenuse of a right-angled isosceles triangle a square ACED is described, and BE meets AC in F. Show that the square on BE is five times the square on a side of the triangle, and that nine times the square on DF is thirteen times the square on the hypotenuse.

PROPOSITION XLVIII. THEOREM.

If the square described on one of the sides of a triangle, be equal to the squares described on the other two sides of it; the angle contained by these two sides is a right angle.

Let the sq. described on BC, one of the sides of △ ABC, = the sqs. on the other two sides AB, AC.

Then shall ∠ BAC be a rt. ∠ .
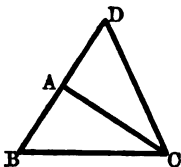
Constr. From A draw AD ⊥ AC, (i. 11), make
            AD = AB, (i. 3), and join DC.
Dem. ∵ AD = AB ∴ sq. on AD = sq. on AB ;
        to each of these =s add the sq. on AC ;
        ∴ sqs. on AD, AC = sqs. on AB, AC :
        but sqs. on AD, AC = sq. on DC,   (i 47)
            ∵ ∠ DAC is a rt. ∠ ;
        and sq. on BC = sqs. on BA, AC ;   (hyp.)
    ∴ sq. on DC = sq. on BC ; ∴ DC = BC.
And ∵ AD = AB, and AC is common to △s DAC, BAC ;
    ∴ AD, AC = AB, AC each to each ; and the base DC
    has been shown = the base BC ; ∴ ∠ DAC = BAC ;
                                            (i. 8)
    but DAC is a rt. ∠ , (constr.); ∴ BAC is a rt. ∠ .
        Therefore, *if the sq. described on*, &c.   Q. E. D.

[This prop. (which is the converse of prop. 47) concludes the third section of Book i., which section relates to equivalence (or equality of area), of figures which are not identical

in form, not equal in all respects. The learner should not leave the first book before he has thoroughly understood and mastered the text : he will do well also to remember the different enunciations by the *numbers* of the propositions, which will not only assist him in calling to mind their *order*, which is frequently useful, but he will thereby be able to give the proper references to preceding propositions in the course of his demonstrations. It is too much to expect that he will at present be able to solve *all* the riders, but some he certainly should do, perhaps with a little assistance here and there.]

BOOK II.

DEFINITIONS.

1. Every right-angled parallelogram is called a *rect-angle*, and is said to be contained by any two of the straight lines which contain one of the right angles.

2. In any parallelogram, one of the parallelograms about a diameter, together with the two complements, is called a *gnomon*.
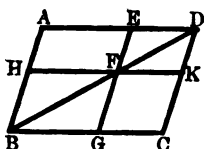
Thus EK with the complements AF, FC is the gnomon AKG.

[Book ii. treats of the properties and relations of rectangles.
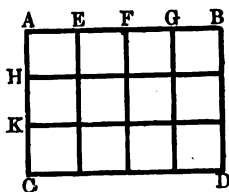
Let AB represent four yards, and AC, perpendicular to it, three yards; complete the rectangle ABDC, and draw perpendiculars to AB, AC through E, H, &c.

The rectangle is now divided into twelve squares and each of their sides is a yard: thus it contains 12 sq. yds., which is said to be its area. Similar reasoning will hold whatever be the linear unit taken and whatever the number of units in

AB, AC. Hence, generally, the area of a rectangle whose sides contain a and b linear units is ab square units. Now it is proved in i. 41 that a triangle is half of any parallelogram (and therefore half of the rectangle) on the same base and between the same parallels. Hence the area of a triangle whose altitude (that is, the perpendicular from the vertex on the base) is a units long, and whose base is b units long, is $\frac{1}{2}ab$.

But it may happen that there is no number, whole or fractional, which will express the number of times which AC contains the unit of length. For instance if AB were the diagonal of a square whose side is equal to AC: then if AB contains a units we should have AC $= \frac{a}{\sqrt{2}}$ (i. 47), a non-terminating decimal equal to no exact number, whole or fractional, and AB, AC have no common measure, that is, they are *incommensurable*. In geometry we frequently meet with incommensurable magnitudes, in which case our method of finding the area of the rectangle AD would fail. Thus we cannot satisfactorily demonstrate the propositions of the second book concerning rectangles and their properties by means of algebraical processes, because, (1) such demonstrations would only hold when the sides of the rectangle were commensurable, and, (2) because the subject of Geometry is not *number*, but *magnitude*: we must prove the propositions by means of geometrical constructions and demonstrations. It may be useful however to give algebraical proofs—(rather, *analogous* or *corresponding* results in algebra)—of the propositions, observing that they do not hold when the lines have no common measure, and they cannot in any case replace the more rigorous, more general, and more elegant geometrical demonstrations given by Euclid.

The only abbreviations admitted for 'the square on AB' is 'sq. on AB,' and for 'the rectangle contained by AB and CD,' 'rect. AB, CD.']

PROPOSITION I.   THEOREM.

If there be two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the rectangle contained by the undivided line, and the several parts of the divided line.

Let AB and CD be two st. lines ; and let CD be divided into any parts in E, F.

Then shall rect. AB, CD = rect. AB, CE, together with rect. AB, EF and rect. AB, FD.
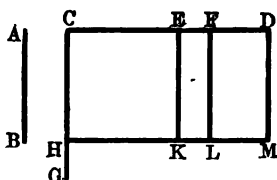
Constr. From C draw CG ⊥ CD, (i. 11) and make CH = AB ; (i. 3) through H draw HM ∥ CD, and through E, F, D draw EK, FL, DM ∥ CH.   (i. 31)

Dem. Then rect. CM = rects. CK, EL, FM.

But CM is contained by AB, CD, for it is contained by CH, CD and CH = AB ; and CK is contained by AB, CE, for it is contained by CH, CE, and CH = AB ; similarly EL is contained by AB, EF, for EK = CH = AB; and FM is contained by AB, FD ;

therefore the rect. AB, CD = rect. AB, CE, together with rect. AB, EF and rect. AB, FD.

Therefore, *if there be,* &c.   Q. E. D.

[The corresponding formula in algebra runs thus :— Let AB, CD be commensurable and let AB contain *a* linear

units, CD b units, and suppose CE, EF, FD contain re-
spectively x, y, z units.   Then $b = x + y + z$: multiply these
equals by a, $\therefore ab = ax + ay + az$; that is, ' if there be two
numbers, one of which is divided into any number of parts,
the product of the two numbers is equal to the sum of the
products of the undivided number and the several parts of
the other.']

PROPOSITION II. THEOREM.

If a straight line be divided into any two parts, the rectangles contained by the whole and each of the parts are together equal to the square on the whole line.

Let the st. line AB be divided into any two parts at C.

Then rect. AB, BC together with rect. AB, AC, = sq. on AB.
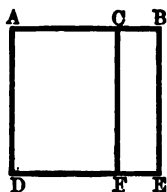
Constr. On AB describe sq. AE, (i. 46) and through C draw CF ∥ AD or BE.

Dem. Then AE = rects. AF, CE.

But AE is the sq. on AB ; AF = the rect. AC, AB,

for it is contained by AC, AD, and AD = AB ;

and CE = rect. AB, CB, for BE = AB ;

∴ rect. AB, AC together with rect. AB, BC = sq. on AB.

Therefore, *if a straight line*, &c.   Q. E. D.

[Corresponding result in algebra :—Let AB contain *b* units, AC, BC, *x*, *y* units respectively; then *b* = *x* + *y*, ∴ $b^2 = bx + by$. The reader should interpret this formula in terms corresponding to the enunciation of the proposition.]

PROPOSITION III. THEOREM.

If a straight line be divided into any two parts, the rectangle contained by the whole and one of the parts, is equal to the rectangle contained by the two parts, together with the square on the aforesaid part.

Let AB be divided into any two parts at C.

Then rect. AB, BC = rect. AC, BC, together with sq. on BC.
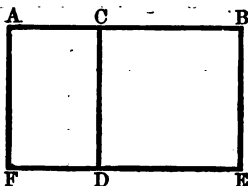
Constr. On BC describe the sq. CE, (i. 46) produce ED, and through A draw AF ∥ CD or BE. (i. 31)

Dem. Then rect. AE = rects. AD, CE;

but AE = rects. AB, BC, for it is contd. by

AB, BE, and BE=BC; also AD = rect. AC, CB,

for CD=CB; and CE is the sq. on BC;

∴ rect. AB, BC = rect. AC, CB together with sq. on BC.

Therefore, *if a straight line,* &c.

[Corresponding result in algebra:—Let AB contain b units; AC, x; BC, y units. Then $b = x + y$ ∴ $by = xy + y^2$. This should be interpreted. A little reflection will show that props. 2 and 3 are particular cases of prop. 1. So in the algebraical formulæ, in the first make $z = o$ and $a = b$, and the second is the result; make $z = o$ and $a = y$, and we get the third. Thus props. 2 and 3 might have been appended to prop. 1 as corollaries, but Euclid considers the cases of sufficient importance to require separate and independent proofs.]

PROPOSITION IV.   THEOREM.

If a straight line be divided into any two parts, the square on the whole line is equal to the squares on the two parts, together with twice the rectangles contained by these parts.

Let AB be divided into any two parts at C.

Then sq. on AB=sqs. on AC and BC, together with twice rect. AC, BC.
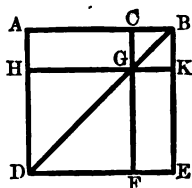
Constr. On AB describe the sq. ADEB, (i. 46) join BD, through C draw CGF ∥ AD or BE, and through G draw HGK ∥ AB or DE.   (i. 31.)

Dem. Then ∵ BD falls on the ∥s CF, AD, ∴ ext. ∠ BGC = int. ∠ ADB ;   (i. 29.)

but ∠ ADB = ∠ ABD ∵ BA = AD,   (i. 5.)

∴ ∠ CGB = ∠ CBG ; ∴ BC = CG ;   (i. 6.)

but CB = GK, and CG = BK ;   (i. 34.)

∴ the figure CK is equilateral.

It is also rectangular, for ∵ CG is ∥ BK, and CB meets them, ∴ ∠s GCB, CBK = two rt. ∠s ;   (i. 29.)

but KBC is a rt. ∠ ; (constr.) ∴ GCB is a rt. ∠ :

∴ also the opp. ∠s CGK, BKG are rt. ∠s; (i. 34.)

∴ CK is rectangular, and it was proved equilateral ;

∴ it is a sq., and it is on BC.

Similarly HF is a sq., and it is on HG, which = AC.

(i. 34.)

Now compl. AG = compl. GE ;   (i. 43.)

and AG = rect. AC, BC, for GC = CB ;

∴ AG, GE together = twice AG, that is, twice rect. AC, BC ; and HF, CK are the sqs. on AC, BC ;

∴ HF, CK, AG, GE = the sqs. on AC, BC, and twice the rect. AC, BC ;

but HF, CK, AG, GE make up the whole figure ABED, which is the sq. on AB ;

∴ the sq. on AB = the sqs. on AC and BC, with twice the rect. AC, BC.

Therefore, *if a straight line,* &c.   Q. E. D.

Cor. It follows from the demonstration, that the parallelograms about the diameter of a square are also squares.

[This is a prop. of great importance, and it should be thoroughly mastered before passing on.   The main thing to be proved is that the figures CK, HF are squares, which might have been done more shortly by applying the corollary to prop. 46.   But Euclid proves again with great clearness that 'if one angle of a parallelogram be a right angle, all its angles will be right angles,' and having established the point that the parallelograms about the diameter of a square are also squares, he refers in subsequent props. (viz. 5, 6, 7) to this corollary, when the same thing has to be shown.

The corresponding algebraical formula is as follows :— Let AB contain a units, and let it be divided into any parts at C, so that AC contains x and BC y units.   Then $a = x + y$ ∴ (squaring both sides) $a^2 = x^2 + y^2 + 2xy$; or, if a number be divided into any two parts, &c.]

1. From the right angle of a right-angled triangle a perpendicular is drawn on the base : show that the square on it is equal to the rectangle contained by the segments of the base.

2. If the square on the perpendicular from the vertex

of a triangle on the base is equal to the rectangle contained by the segments of the base, the vertical angle is a right angle.

3. From one of the equal angles of an isosceles triangle a perpendicular is drawn to the opposite side : show that the square on the base is equal to twice the rectangle contained by that side and the part of it between the perpendicular and the base.

4. From one of the equal angles of an isosceles triangle a perpendicular is drawn to the opposite side : show that the square on the perpendicular is equal to twice the rectangle contained by the segments of that side, together with the square on the line intercepted between the other equal angle and the perpendicular.

PROPOSITION V. THEOREM.

If a straight line be divided into two equal parts, and also into two unequal parts, the rectangle contained by the unequal parts, together with the square on the line between the points of section, is equal to the square on half the line.

Let AB be divided into two equal parts at C, and into two unequal parts at D.

Then rect. AD, DB, together with sq. on CD = sq. on CB.
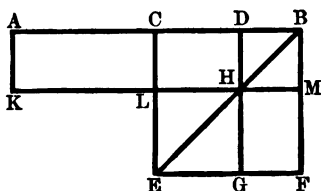
Constr. On CB describe sq. CF, (i. 46) join BE, through D draw DHG ‖ CE or BF, through H draw KLM ‖ CB or EF, and through A draw AK ‖ CL or BM.

Dem. Then, ∵ compl. CH = compl. HF, (i. 43) add to each of these = s DM ; ∴ the whole CM = the whole DF; but ∵ AC = CB, ∴ AL = CM, (i. 36) ∴ also AL=DF; add to each of these equals CH, and the whole AH = CH and DF ; that is, rect. AD, DB = the gnomon CMG ;

to each of these equals add LG, which = sq. on CD ;

(ii. 4. cor.)

∴ rect. AD, DB together with sq. on CD = gnomon CMG together with LG:

but gnomon CMG with LG make up the whole figure CF, which is sq. on CB.

∴ rect. AD, DB, together with sq. on CD = sq. on CB.

Therefore, *if a straight line*, &c. Q. E. D.

Cor. From this proposition it follows, that the difference of the squares on two unequal lines is equal to the rectangle contained by their sum and their difference.

[The corollary follows thus :—Let AC, CD be two lines of which AC is the greater; produce AC to B, making CB = AC. Then, ∵ AB is divided into two equal parts in C and into two unequal parts in D, rect. AD, DB together with sq. on CD = sq. on AC ; take away from each of these = s sq. on CD, and rect. AD, DB = difference of sqs. on AC and CD.

Algebraically. Let AB contain $2a$ linear units, and CD b units. Then AD contains $a + b$, and DB $a - b$ units ; now $(a + b)(a - b) = a^2 - b^2$: add b^2 to each, and $(a + b)(a - b) + b^2 = a^2$; that is, if a number be divided into two equal parts, &c.]

AD is drawn from the vertex of an isosceles triangle to any point D in the base BC. Show that the difference of the squares on AB, AD is equal to the rectangle contained by BD, CD.

PROPOSITION VI. THEOREM.

If a straight line be bisected and produced to any point, the rectangle contained by the whole line thus produced and the part of it produced, together with the square on half the line bisected, is equal to the square on the line which is made up of the half and the part produced.

Let AB be bisected in C and produced to D.

Then rect. AD, DB, together with sq. on BC, = sq. on CD.
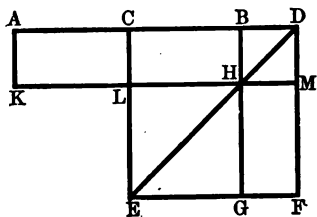
Constr. On CD describe sq. CF, (i. 46.) and join DE, through B draw BHG ∥ CE or DF, through H draw KLM ∥ AD or EF, and through A draw AK ∥ CL or DM.   (i. 31)

Dem. Then, ∵ AC = CB, ∴ AL = CH ;   (i. 36)

  but CH = HF ; (i. 43) ∴ AL = HF ; to each add CM ;
            ∴ AM = gnomon CMG :

but AM is rect. AD, DB, for DM = DB ;

∴ gnomon CMG = rect. AD, DB : to each add LG,
    which = sq. on CB ;   (ii. 4, cor.)

∴ rect. AD, DB with sq. on CB = gnomon CMG and
    LG ; but gnomon CMG and LG make up the whole
    figure CF, which is sq. on CD ;

∴ rect. AD, DB, together with sq. on CB = sq. on CD.

Therefore, *if a straight line*, &c. Q. E. D.

[The corresponding result is thus shown algebraically :—
Let AB contain $2a$ linear units, then BC contains a units;
let BD contain x units, then AD contains $2a + x$, and since
$x(2a + x) = 2ax + x^2$; ∴ (adding a^2 to each), $a^2 + x(2a + x)$
$= a^2 + 2ax + x^2$, $= (a + x)^2$; that is, if a number be divided,
&c.]

PROPOSITION VII. THEOREM.

If a straight line be divided into any two parts, the squares on the whole line, and on one of the parts, are equal to twice the rectangle contained by the whole and that part, together with the square on the other part.

Let AB be divided into any two parts at C.

Then the sqs. on AB, BC = twice rect. AB, BC together with sq. on AC.
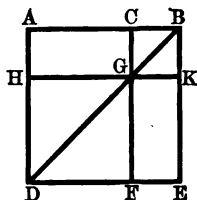
Constr. On AB describe sq. AE, (i. 46) join BD, through C draw CF ∥ AD or BE, cutting BD at G, and through G draw HGK ∥ AB or DE.   (i. 31)

Dem. Then, ∵ AG = GE, (i. 43) add CK to each ;
∴ AK = CE, and ∴ AK, CE are double of AK ;
but AK, CE are the gnomon AKF and sq. CK ;
∴ gnomon AKF and sq. CK are double of AK ;
but twice rect. AB, BC is double of AK, for BK = BC ;
∴ gnomon AKF and sq. CK = twice rect. AB, BC ;
          add to each HF, which = sq. on AC ;
                              (ii. 4, cor.)
∴ gnomon AKF, and sqs. CK, HF = twice rect. AB, BC and sq. on AC ;
but gnomon AKF and sqs. CK, HF make up the whole figure AE and CK, which are the sqs. on AB, BC ;

∴ sqs. on AB and BC = twice rect. AB, BC together with sq. on AC.

Therefore, *if a straight line*, &c. Q. E. D.

[Algebraical form :—Let AB contain *a* units, and let it be divided into any two parts at C; let AC contain *x* units; then BC contains $a-x$ units. Now $(a-x)^2 = a^2 - 2ax + x^2$, ∴ $a^2 + (a-x)^2 = 2a^2 - 2ax + x^2 = 2a(a-x) + x^2$; that is, if a number be divided into any two parts, &c.]

1. From the right angle of a right-angled triangle a perpendicular is drawn on the hypotenuse: show that the square on each of the sides is equal to the rectangle contained by the hypotenuse and the segment of it adjacent to that side.

2. If a perpendicular be drawn from the vertical angle of a triangle on the base, and the square on each side be equal to the rectangle contained by the base and the segment of it adjacent to that side, then the vertical angle is a right angle.

3. Show that the sum of the squares on two straight lines is never less than twice their rectangle.

4. Produce a given line so that the rectangle of the whole line produced and the given line shall be equal to a given square.

Is this always possible?

PROPOSITION VIII. THEOREM.

If a straight line be divided into any two parts, four times the rectangle contained by the whole line and one of the parts, together with the square on the other part, is equal to the square on the straight line which is made up of the whole and that part.
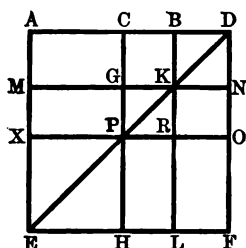
Let AB be divided into any two parts at C.

Then four times rect. AB, BC with sq. on AC = sq. on the line made up on AB and BC together.

Constr. Produce AB to D making BD = CB, (i. 3) on AD describe sq. AF, and complete the figures as in the preceding propositions.

Dem. ∵ CB = BD, and CB = GK, BD = KN,
∴ GK = KN ; similarly PR = RO ;
hence, ∵ CB = BD, and GK = KN,
∴ rect. CK = rect. BN, and GR = RN ; (i. 36)
but compl. CK = compl. RN, (i. 43) ∴ BN = GR ;
∴ the four rects. BN, CK, GR, RN are = one another, and are together quadruple one of them, CK.

Again, ∵ CB = BD, BD = BK, that is = CG ;
and ∵ CB = GK, that is = GP; ∴ CG = GP.

And ∵ CG = GP, and PR = RO, ∴ rect. AG = MP,
and PL = RF ;

but compl. MP = compl. PL, (i. 43) ∴ AG = RF ;

∴ also the four rects. AG, MP, PL, RF are = one
another, and are together quadruple of one of them,
AG.

And it was shown that CK, BN, GR, RN are quadruple
of CK ; ∴ the eight rects. which make up the gnomon
AOH are quadruple of AK.

But AK is rect. AB, BC, ∵ BK = BC; ∴ four times
rect. AB, BC is quadruple of AK, and is therefore equal
to the gnomon AOH.   (ax. 1)

To each add XH, which = sq. on AC ;

∴ four times rect. AB, BC, with sq. on AC, = gnomon
AOH and sq. XH, which together make up the figure
AF, the sq. on AD ;

∴ four times rect. AB, BC together with sq. on AC = sq. on
AD, that is on the line made up of AB and BC together.

Therefore, *if a straight line,* &c.   Q. E. D.

[Corresponding result in algebra:—Let AB which contains
a units be divided at C, and let AC contain *x* units ; then
since $(a-x)^2 = a^2 - 2ax + x^2$, add $4ax$ to both sides and
$4ax + (a-x)^2 = a^2 + 2ax + x^2 = (a+x)^2$. Now BC contains $a-x$
units, hence the analogous result is established.]

PROPOSITION IX. THEOREM.

If a straight line be divided into two equal, and also into two unequal parts, the squares on the two unequal parts are together double of the square on half the line, and of the square on the line between the points of section.

Let the st. line AB be divided into two equal parts at C, and into two unequal parts at D.

Then the sqs. on AD, DB shall be together double of the sqs. on AC, CD.
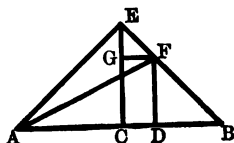
Constr. From C draw CE ⊥ AB, (i. 11) make CE = AC or CB, (i. 3) join EA, EB; through D draw DF ∥ CE, meeting EB in F, through F draw FG ∥ AB, and join AF.

Dem. ∵ AC = CE, ∴ ∠ EAC = AEC ; (i. 5) and ∵ ∠ ACE is a rt. ∠ ∴ ∠ s CEA, CAE are together = a rt. ∠ ;   (i. 32)

but they are = one another, ∴ each of them is half a rt. ∠.

Similarly each of the ∠ s CEB, EBC is half a rt. ∠ ;
                    ∴ ∠ AEB is a rt. ∠.

Again, ∵ GEF is half a rt. ∠, and EGF a rt. ∠, for it is = int. ∠ ECB, (i. 29) ∴ rem. ∠ EFG is half a rt. ∠, ∴ ∠ GEF = ∠ EFG, ∴ EG = GF.   (i. 6)

Again, ∴ ∠ABE is half a rt. ∠, and ∠FDB is a rt. ∠, for it is = int. ∠ECB, (i. 29) ∴ rem. ∠BFD is half a rt. ∠, ∴ ∠DBF = ∠BFD, ∴ DF = DB.   (i. 6)

And ∵ AC = CE, ∴ sq. on AC = sq. on CE, ∴ sqs. on AC, CE are together double of sq. on AC ;

but sq. on AE is = sqs. on AC, CE ;   (i. 47)
            ∴ sq. on AE is double of sq. on AC.

Again, ∵ EG = GF, ∴ sq. on EG = sq. on GF, ∴ sqs. on EG, GF are together double of sq. on GF ;

but sq. on EF = sqs. on EG, GF ;   (i. 47)
            ∴ sq. on EF is double of sq. on GF ;

and GF = CD, (i. 34) ∴ sq. on EF is double of sq. on CD.

Now sq. on AE is double of sq. on AC, ∴ sqs. on AE, EF are double of sqs. on AC, CD ;
            but sq. on AF = sqs. on AE, EF ;   (i. 47)
        ∴ sq. on AF is double of sqs. on AC, CD ;
            but sqs. on AD, DF = sq. on AF ;   (i. 47)
        ∴ sqs. on AD, DF are double of sqs. on AC, CD ;
                    and DF = DB ;
        ∴ sqs. on AD, DB are double of sqs. on AC, CD.
            Therefore, *if a straight line,* &c.   Q. E. D.

[The demonstration is long, though its different steps are simple ; it is shown (1) ∠AEB is a rt. ∠, (2) EG = GF, (3) DF = DB, (4) sq. on AE is double sq. on AC, (5) sq. on EF is double sq. on CD, (6) sq. on AF is double of sqs. on AC, CD, (7) sqs. on AD, DB are double sqs. on AC, CD.

The corresponding formula in algebra may thus be obtained. Let AB contain $2a$ units and let CD contain x units, so that AD contains $a+x$, and DB, $a-x$ units. Now $(a+x)^2 = a^2 + 2ax + x^2$; $(a-x)^2 = a^2 - ax + x^2$; ∴ adding, $(a+x)^2 + (a-x)^2 = 2a^2 + 2x^2$: that is, *if a number be divided,* &c.]

1. The square on the sum of two straight lines to-

gether with the square on their difference is double the squares on the two lines.

  2. Divide a straight line into two parts so that the sum of their squares may be the least possible.

  3. ABCD is a square and AC its diagonal. In AC a point E is taken. Show that the triangle, whose sides are equal to AE, EC, and the diagonal of a square described on BE, will contain a right angle.

PROPOSITION X. THEOREM.

If a straight line be bisected, and produced to any point, the square on the whole line thus produced, and the square on the part of it produced, are together double of the square on half the line bisected, and of the square on the line made up of the half and the part produced.

Let AB be bisected in C and produced to D.

Then shall sqs. on AD, DB be double sqs. on AC, CD.
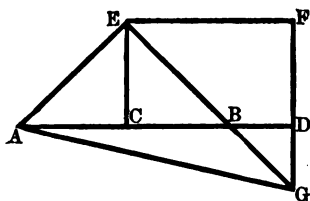
Constr. From C draw CE ⊥ AB, (i. 11), make CE = AC or CB, (i. 3) join AE, EB ; through E draw E F ∥ AB, and through D draw DF ∥ CE.

Then ∵ EF meets ∥ s CE, FD, ∴ int ∠ s CEF, EFD = two rt. ∠ s ;   (i. 29)

  ∴ ∠ s BEF, EFD are less than two rt. ∠ s, and

    ∴ EB, FD will meet, if produced ;   (ax. 12)

    let them meet in G, and join AG.

Dem. ∴ AC = CE, ∴ ∠ CAE = ∠ CEA ;   (i. 5)

  and ∠ ACE is a rt. ∠ ; ∴ each of ∠ s CAE, CEA is half a rt. ∠ .   (i. 32)

    Similarly, each of ∠ s CEB, CBE is half a rt. ∠ .

            ∴ ∠ AEB is a rt. ∠ .

And ∵ EBC is half a rt. ∠ , ∴ DBG is also half a rt. ∠ ;

                        (i. 15)

but $\angle$ BDG is a rt. $\angle$, being $=$ alt. $\angle$ DCE;   (i. 29)

$\therefore$ rem. $\angle$ DGB is half a rt. $\angle$, and is $\therefore$ $=$ $\angle$ DBG;

$$\therefore \text{ BD} = \text{DG.} \qquad\qquad \text{(i. 6)}$$

Again, $\because$ EGF is half a rt. $\angle$, and $\angle$ EFD is a rt. $\angle$,

being $=$ the opposite $\angle$ ECD,   (i. 34)

$\therefore$ rem. $\angle$ FEG is half a rt. $\angle$, and is $\therefore$ $=$ $\angle$ EGF;

$$\therefore \text{ GF} = \text{FE.} \qquad\qquad \text{(i. 6)}$$

And $\because$ EC $=$ CA, $\therefore$ sq. on EC $=$ sq. on CA; $\therefore$ sqs. on EC, CA are double of sq. on CA;

but sq. on EA $=$ sqs. on EC, CA; $\therefore$ sq. on EA is double of sq. on CA.

And $\because$ GF $=$ EF, $\therefore$ sq. on GF $=$ sq. on EF, $\therefore$ sqs. on GF, FE are double of sq. on EF;

but sq. on EG $=$ sqs. on GF, FE; $\therefore$ sq. on EG is double of sq. on FE;

and EF $=$ CD, (i. 34) $\therefore$ sq. on EG is double of sq. on CD;

and it was shown that sq. on EA is double of sq. on CA;

$\therefore$ sqs. on EA, EG are double of sqs. on AC, CD;

but sq. on AG $=$ sqs. on EA, EG;   (i. 47)

$\therefore$ sq. on AG is double of sqs. on AC, CD;

but sqs. on AD, DG $=$ sq. on AG; $\therefore$ sqs. on AD, DG are double of sqs. on AC, CD;

but DG $=$ DB;

$\therefore$ sqs. on AD, DB are double of sqs. on AC, CD.

Therefore, *if a straight line*, &c.   Q. E. D.

[The demonstration corresponds closely with that of the preceding proposition, and the successive steps are these :— (1) $\angle$ AEB is a rt. $\angle$; (2) BD $=$ DG; (3) GF $=$ FE; (4) sq. on EA is double sq. on CA; (5) sq. on EG is double sq. on CD; (6) sq. on AG is double sqs. on AC, CD; (7) sqs. on AD, DB are double sqs. on AC, CD.

The corresponding algebraical result is as follows :—Let AB contain $2a$, and BD x units. Now $(2a + x)^2 = 4a^2 + 4ax + x^2$; add x^2 to each of these $=$ s, and $(2a + x)^2 + x^2 = 4a^2 + 4ax + 2x^2$

$= 2a^2 + 2(a^2 + 2ax + x^2) = 2a^2 + 2(a + x)^2$; that is, if a number be divided, &c.

The proper construction for the solution of a geometrical problem may often be *indicated* by employing algebraical symbols. The student who has been able to solve a fair number of the riders up to this point may with advantage consider the following problem, illustrating this remark: ' To divide a given line into two parts so that the square on one part may be double the square on the other.' Let the length of the line AB be a, suppose it be divided as required in D, and let DB $= x$. Then since sq. on AD $=$ twice sq. on BD $\therefore (a - x)^2 = 2x^2$. A solution of this quadratic is $x = a(\sqrt{2} - 1)$. Now $a\sqrt{2}$ indicates the diagonal of the sq. on AB (for twice the sq. on a side $=$ sq. on the diagonal, by i. 47), hence the following construction is suggested by the solution. Produce AB to C making BC $=$ AB. On BC describe a square, and with centre C, radius $=$ diagonal of sq., describe a circle cutting AB in D : D is the point required.

For by prop. 9, sq. on AD with sq. on DC are double sqs. on DB and BC. But sq. on DC is double sq. on BC, by constr., $\therefore$ sq. on AD is double sq. on DB. Again, take this problem : ' To produce a given straight line so that the square on the whole line thus produced may be double the square on the part produced.' As before, let AB $= a$, BD $= x$, then since sq. on AD $=$ twice sq. on BD, $(a + x)^2 = 2x^2$. This quadratic solved gives $x = a(\sqrt{2} + 1)$ as one solution, and this value suggests the construction, namely : produce AB to C making BC $=$ AB ; on BC describe a square : with centre C and radius $=$ diagonal of sq. describe a circle, cutting *AC produced* in D : D is the pt. required. For by prop. 10, sqs. on AD, DC are double sqs. on DB, BC. But sq. on DC ($=$ sq. on diagonal) is double sq. on BC, $\therefore$ sq. on AD is double sq. on BD.

The thoughtful reader will have observed a striking correspondence between the solutions of these two problems and the indicating equations, and he will also have noticed that each of the quadratics has two solutions. We cannot dwell further on this remarkable point at present : the student may return to it with great profit at a more advanced stage of his reading. It is sufficient to note the use of algebraical symbols in *suggesting* the solution of problems of this nature. Observe, in *suggesting* only :—see the note at the beginning of Book ii.]

PROPOSITION XI.   PROBLEM.

To divide a given straight line into two parts, so that the rectangle contained by the whole and one of the parts may be equal to the square on the other part.

Let AB be the given st. line.   It is required to divide AB into two parts, so that rect. contained by AB and one part may = sq. on the other part.
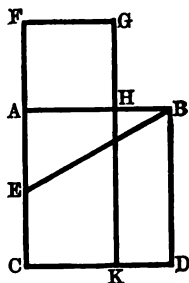
Constr. On AB describe sq. AD ; (i. 46) bisect AC at E, (i. 10) and join BE ;

produce CA to F, make EF = EB, and on AF describe sq. FH.

Then shall AB be divided at H so that rect. AB, BH = sq. on AH.

Produce GH to meet CD in K.

Dem. ∵ AC is bisected in E and produced to F, ∴ rect. CF, FA with sq. on AE = sq. on EF ;   (ii. 6)

but EF = EB, ∴ rect. CF, FA with sq. on AE = sq. on EB ;

but sqs. on BA, AE = sq. on EB, (i. 47);

∴ rect. CF, FA with sq. on AE = sqs. on BA, AE ;

take away sq. on AE from each of these = s ;

∴ rect. CF, FA = sq. on BA.

But FK is rect. CF, FA, for FA = FG ; and AD is sq.
  on AB ;

$$\therefore FK = AD ;$$

take away the common part AK,

$$\therefore \text{rem. } FH = \text{rem. } HD ;$$

but HD = rect. AB, BH, for AB = BD ;

and FH is sq. on AH ;

$$\therefore \text{rect. } AB, BH = \text{sq. on } AH,$$

Therefore, *the given straight line is divided as required.*

Q. E. F.

1. Show that the squares on the whole line and one
part are equal to three times the square on the other
part.

2. Show that the rectangle contained by the sum and
difference of the parts is equal to the rectangle contained
by the parts.

3. If CH produced meets BF at L, the angle CLB
is a right angle.

PROPOSITION XII. THEOREM.

In obtuse-angled triangles, if a perpendicular be drawn from either of the acute angles to the opposite side produced, the square on the side subtending the obtuse angle is greater than the squares on the sides containing the obtuse angle, by twice the rectangle contained by the side on which, when produced, the perpendicular falls, and the straight line intercepted without the triangle, between the perpendicular and the obtuse angle.

Let ABC be an obtuse-angled △, having the obtuse ∠ ACB, and from A let AD be drawn ⊥ BC produced.

Then sq. on AB shall be greater than sqs. on AC, CB, by twice the rect. BC, CD.
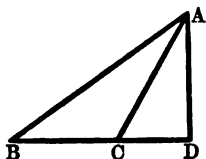
Dem. ∵ BD is divided into two parts at C, ∴ sq. on BD = sqs. on BC, CD, and twice rect. BC, CD;

(ii. 4)

to each of these = s add sq. on DA ;

∴ sqs. on BD, DA = sqs. on BC, CD, DA and twice rect. BC, CD ;

but sq. on BA = sqs. on BD, DA, and sq. on AC = sqs. on CD, DA ;

(i. 47)

∴ sq. on BA = sqs. on BC, AC, and twice rect. BC, CD; that is, sq. on BA is greater than sqs. on BC, AC, by twice rect. BC, CD.

Therefore, *in obtuse-angled triangles,* &c. Q. E. D.

1. The perpendiculars from two angles A, B of an equilateral triangle on the opposite sides intersect in O : show that the square on AO is a third of the square on AB.

2. In the triangle ABC each of the angles at B, C is double the angle at A : show that the square on AB is equal to the square on BC together with the rectangle AB, BC.

PROPOSITION XIII. THEOREM.

In every triangle, the square on the side subtending an acute angle, is less than the squares on the sides containing that angle, by twice the rectangle contained by either of these sides, and the straight line intercepted between the perpendicular let fall on it from the opposite angle, and the acute angle.

Let ABC be any $\triangle$, and $\angle$ B one of its acute $\angle$s, and upon BC, produced if necessary, let fall the $\perp$ AD from the opp. $\angle$; then sq. on AC opp. $\angle$ B shall be less than sqs. on CB, BA, by twice rect. CB, BD.
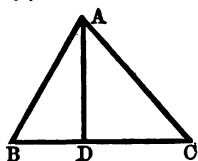
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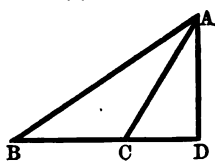
Dem. For in fig. 1, BC is divided into two parts in D, and in fig. 2, BD is divided into two parts in C,

$\therefore$ in both, sqs. on BC, BD = twice rect. BC, BD and sq. on CD. (ii. 7)

Add to each of these =s sq. on DA,

then sqs. on BC, BD, DA = twice rect. BC, BD and sqs. on CD, DA ;

but sq. on AB = sqs. on BD, DA ; sq. on AC = sqs. on CD, DA ; (i. 47)

$\therefore$ sqs. on BC, AB = twice rect. BC, BD and sq. on AC ;

$\therefore$ sq. on AC is less than sqs. on AB, BC by twice rect. BC, BD.

The case when AC is ⊥ BC needs no proof.   (i. 47)

Therefore, *in every triangle*, &c.   Q. E. D.

1. DE is parallel to the base BC of an isosceles triangle ABC. Show that the sq. on BE is equal to the rectangle contained by BC, DE, together with the sq. on CE.

2. If D be a point in the side BC of an equilateral triangle ABC prove that the squares on BD, CD are together less than the square on AD by the rectangle contained by BD, CD.

PROPOSITION A. THEOREM.

In every triangle, the sum of the squares on two of the sides, is equal to twice the square on half the base, together with twice the square on the line drawn from the vertex to the middle point of the base.

Let ABC be △ and let D be the middle point of the base AB. Then shall the sqs. on AC, BC = twice the sq. on AD and twice the sq. on CD.
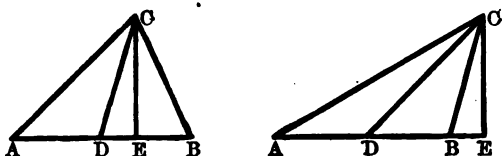
Constr. Draw CE ⊥ AB.

Dem. Of the two ∠s ADC, BDC one must be acute and the other obtuse, or each is a rt. ∠.

The case when each is a rt. ∠ needs no proof. (i. 47)

Let ∠ ADC be an obtuse ∠.

Then sq. on AC = sqs. on AD, DC with twice rect. AD, DE ;   (ii. 12)

sq. on BC with twice rect. BD, DE = sqs. on BD, DC ;   (ii. 13)

∴ sqs. on AC, BC, with twice rect. BD, DE = sqs. on AD, DB, and twice sq. on DC and twice rect. AD, DE ;   (ax. 2)

but rect. BD, DE = rect. AD, DE, and sqs. on AD, DB = twice sq. on AD,

$$\because \text{AD} = \text{DB} ; \qquad \text{(constr.)}$$

∴ sqs. on AC, BC = twice sq. on AD and twice sq. on DC.

        Therefore, *in every triangle*, &c.   Q. E. D.

[This proposition, which is deduced directly from props. 12, 13, is not given by Euclid, but it seems to deserve a place in the second Book, not only on account of its own interest, but because also it is the key to a large class of problems, which involve by the enunciations the bisection of some line. Some examples of such problems will now be given.]

1. If straight lines be drawn from each angle of a triangle to bisect the opposite sides, four times the sum of the squares on these lines is equal to three times the sum of the squares on the sides.

2. If AB, one of the equal sides of an isosceles triangle ABC, be produced beyond the base to D, so that BD = AB, show that the square on CD is equal to the square on AB together with twice the square on BC.

3. From any point P lines are drawn to the angles of a rectangle ABCD ; show that the squares on PA, PC are together equal to the squares on PB, PD.

4. A square BDEC is described on the hypotenuse of a right-angled triangle ABC ; show that the squares on DA, AC are together equal to the squares on EA, AB.

5. The squares on the diagonals of a parallelogram are together equal to the squares on the four sides.

6. If two points be taken in the diameter of a circle equally distant from the centre, the sum of the squares of two lines drawn from these points to any point in the circumference will be constant.

7. The hypotenuse of a right-angled triangle ABC is trisected at D, E, and CD, CE are joined ; show that the sum of the squares on the sides of the triangle CDE is equal to two-thirds of the square on AB.

8. In any quadrilateral, the squares on the diagonals are together double of the sum of the squares on the two lines joining the bisections of the opposite sides.

PROPOSITION XIV.   PROBLEM.

To describe a square that shall be equal to a given recti-lineal figure.

Let A be the given rectil. figure.

It is required to describe a square that shall be equal to A.
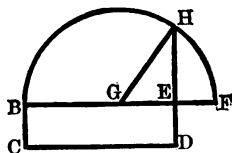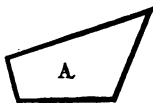
Constr. Describe the rectangular ▱ BCDE = the rectil. figure A. (i. 45)

Then if BE = ED, it is a sq., and what was required is done.

But if BE be not = ED, produce BE to F, and make EF = ED, bisect BF in G, (i. 10) from centre G, at distance GB or GF, describe semicircle BHF, and produce DE to meet the circumference in H.

The sq. on EH shall equal the given figure A.

Dem. ∵ BF is divided into two equal parts at G and into two unequal parts at E, ∴ rect. BE, EF with sq. on EG = sq. on GF ; (ii. 5)

but GF = GH, and ∴ sq. on GF = sq. on GH, and ∴ = sqs. on GE, EH ; (i. 47)

∴ rect. BE, EF, with sq. on EG = sqs. on EG, EH ; take away from each the sq. on EG ;

and rect. BE, EF = sq. on EH.

But rect. BE, EF is BD ∵ EF = ED, (constr.)

∴ BD = sq. on EH ; and BD = A ; (constr.)

∴ sq. on EH = rectil. fig. A.

Therefore, *a square has been described as required.*

Q. E. F.

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